In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institution shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the Dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

THE DESIGN OF AUTOMATIC CONTROL SYSTEMS BASED ON ROOT LOCATION

A THESIS

Presented to

The Faculty of the Graduate Division

by

Albert Tillman Scott

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

Georgia Institute of Technology

December, 1964

THE DESIGN OF AUTOMATIC CONTROL

SYSTEMS BASED ON ROOT LOCATION

Approved:

____

____

____

Date approved by Chairman:

April 16, 1965

ACKNOWLEDGMENTS

The author wishes to express his appreciation to those people who have aided him in this study. To Dr. Eugene Harrison of the School of Mechanical Engineering for suggesting the problem and for continuous guidance, to Dr. Harold Johnson and Dr. R. P. Webb for their critical review and service on the reading committee, and to Jay Schlag for his operation of the analog computer, go the author's heartfelt thanks.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

This study concerns the design of automatic control systems based on the location of the roots of the characteristic equation. The investigation was restricted to linear proportional control systems where the transfer function of the controlled system contains only a constant in the numerator.

The effects of feedback gains on root locations for systems of this type were determined, and the results incorporated in a set of design curves. With these curves a designer may select gains for a high order system, and consequently root locations, so that the system response approximates that of a selected second order system defined by damping ratios ranging from 0.4 to 1.0 and with natural frequencies varying from 2.0 to 20.0 radians per second. The relationships between these two parameters and the dynamic performance of a second order system are well known. Thus, by selecting gains to approximate second order system behavior, the designer may achieve a desired system performance.

Both analog and digital computers were used to obtain the design information, and the circuits and programs necessary are also presented.

The work of Elgerd, D'Azzo, and Mitrovic is included to acquaint the reader with the work of those in this field who have made major contributions.

The results of the study show that the effects of the various feedback parameters (position, rate, acceleration) changes with system order, with the position element dominating as the system order increases. Also

evident is the fact that the need for moving the extra roots further out the negative real s plane axis to more nearly approximate a second order system increases with the order of the actual system.

CHAPTER I

INTRODUCTION

At the present time, few concrete rules for the design of automatic control systems with characteristic equations of order greater than two exist. The purpose of this study is to aid the system designer by presenting information that will enable him to rapidly choose feedback gain constants to achieve a desired system performance. This information is presented in the form of graphs which facilitate rapid determination of these feedback gain constants.

Fig. 1 illustrates a basic closed-loop control system of the type considered in this study.

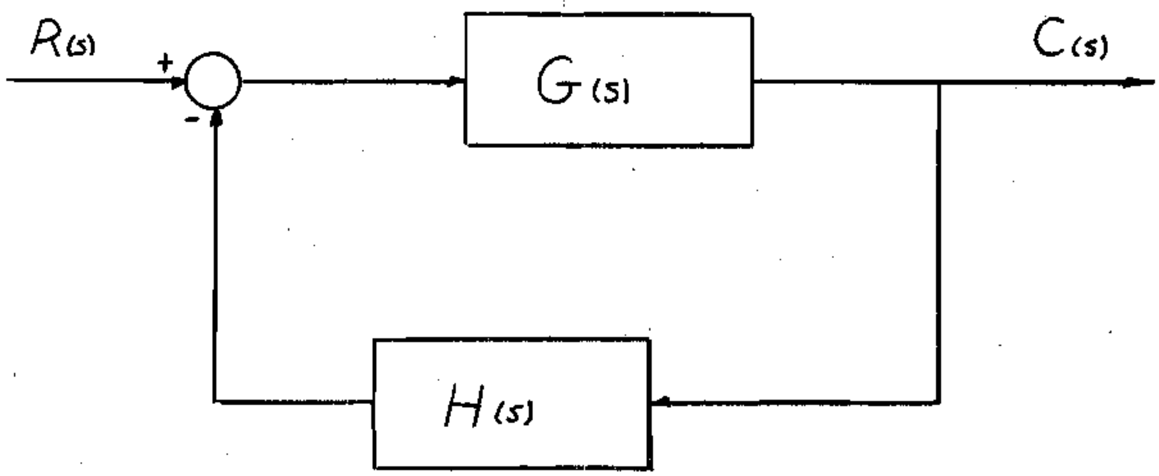


Figure 1. Basic Feedback System

The characteristic equation of the system shown is

$$1 + G(s)H(s) = 0 \tag{1}$$

and the ratio of output to input is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{2}$$

Here G(s) and H(s) represent the feedforward and feedback transfer functions, respectively, C(s) represents output, and R(s) designates the reference input. Given a $G(s) = \frac{1}{s^2 + as + b}$ and $H(s) = K_2 s + K_1$, the characteristic equation using the form of Equation 1 is

$$s^2 + (a + K_2)s + (b + K_1) = 0 \tag{3}$$

A feedforward gain of unity was selected for this example and will be used for all further systems considered in order to simplify the presentation of results. This can be justified because the characteristic equation of a system with non-unity gain K, is the same as that of a system with unity gain and feedback gains equal to KK_1, KK_2, etc.

Smith[1] established relationships between the roots and coefficients of equations such as Equation 3. These relationships for equations up through degree five are displayed in Appendix D.

The reader will note the author has subscribed to the more or less standard procedure of expressing transfer functions with Laplace operator notation in lieu of actual derivative terms. The relationship between a function of time f(t), and its corresponding Laplace transform is defined by

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{4}$$

where

$$s = \nabla + j\omega \tag{5}$$

This procedure has many advantages[2] not the least of which are that boundary conditions are included automatically, and both the transient and steady state components of the solution are obtained simultaneously.

It is shown in Appendix D how root location affects the coefficients of the characteristic equation, but it is not evident how root location affects system performance.

There are many ways of specifying system performance. Ordinarily the designer will want a system to react quickly, accurately, and with little oscillation to a given input. That is to say, he will specify a fast response, short settling time, little steady-state error, and a minimum amount of overshoot. Gibson[3] has made an attempt to standardize system performance specifications. He recommends frequency domain specifications of M-peak, peak frequency, bandwidth, and peak out impedance. For time domain specifications he recommends time delay, rise time, settling time, per cent overshoot, and final value of error. These recommended terms are defined in Appendix C.

Basically, a closed-loop system will be stable (i.e., eventually the output will approach a bounded function of the referenced input) if the roots of the characteristic equation lie in the left-hand half of the s plane. The number, type, and location of these roots in that left-hand plane determine the system's relative stability. If all the roots of the characteristic equation lie in the left-hand plane, all inverse Laplace transforms of the characteristic roots will contain decaying exponentials which diminish with time[4]. It is thus important that the design engineer be able to obtain information regarding the location of each root with minimum effort.

Higher order systems are often specified in terms of an equivalent second order system, and thus these two "control roots" of the higher order system are those which dominate the transient response.

Although all characteristic equation roots have some effect on system response, the significance of any particular root or pair of complex conjugate roots, depends upon the magnitude and rate of exponential decay of the corresponding term in the system output-response function. Roots near the imaginary axis (small negative real parts) correspond to low frequency response or slow decay, while roots far to the left (large negative real parts) represent high-frequency response or rapid exponential decay. A summary of root position and the corresponding predictable response may be found in Appendix E.

It is evident that the characteristic equation is appropriately named in that it does indeed characterize the transient response of the closed-loop control system. A designer may then, by selecting a damping ratio and natural frequency (defined in Appendix C) define a second order system with the desired system performance characteristics. He may then design his higher order actual system so that its behavior approximates that of this chosen second order system by moving the extra roots far to the left of the imaginary axis and thereby make negligible their effect on the actual system response. This is the basis upon which the presented design method was developed.

CHAPTER II

LITERATURE SURVEY OF PREVIOUS WORK

A great deal of work has been done in control system design with regard to root locations, particularly in the areas of system synthesis by root loci, compensation, graphical solutions, and existing system testing. While there is overlapping in these areas the author will enumerate and describe the work done by the leading researchers and indicate such overlap.

There are two basic methods of predicting and analyzing a control system's response: the root locus method and the frequency response method. Each method in its own right is important to the designer for the information that method will yield. "The root locus is a plot of the roots of the characteristic equation as a function of the gain"[5] and indicates the actual time response. The frequency response method yields information concerning the system response at given frequencies, and indicates the changes that should be made to the system to achieve a desired output. However, the frequency response method does not require the determination of the roots of the characteristic equation and for the purpose of this study this method and its development are not considered.

The root locus method developed by Evans[6] and amplified by many others, obviates the long tedious and inherent difficulty of continually solving the system equation for its roots to determine the system's stability characteristics. It presents a graphical display of all closed-loop pole positions for an open-loop gain of from zero to infinity. Steiglitz[7], Jawor[8], Matkowsky[9], and Elgerd[10], have made significant contributions to this method. Steiglitz presents a superposition theorem which shows how the root loci for two open-loop functions place constraints on the locus for their product, and how a knowledge of the simple lower-ordered loci can be used to sketch the resulting loci. Jawor concerns himself mainly with the practical applications of root locus methods to closed-loop systems. Matkowsky used the root locus technique to develop a means of ascertaining certain properties of transient responses. He prescribes conditions for the existence of time functions that bound the unit impulse response of the given system function. Elgerd shows that the time response is uniquely determined by the location of the closed-loop poles and zeros in the s plane, and he presents graphs to be used by a system designer to enable him to correlate the connection between the pole-zero configuration and the real time response.

Mitrovics'[11] extensive work in the field of control system compensation has contributed considerably toward the development of design methods based on root location. He considers the frequency response and transient response simultaneously, but still permits the design process to be effected in the real domain. His method approaches the problem algebraically, thus handling it directly in the form as obtained after applying the Laplace transformation. With his method a designer may (1) effect all operations in the real domain, (2) adjust the coefficients of the characteristic equation without calculations and (3) estimate almost all roots of the characteristic equation simultaneously.

Mariotti[12] in his M.S. thesis at the University of California, presented a direct method of compensation for linear feedback systems. The method is good for systems of any order with an arbitrary number and location of open-loop poles and zeros. His method allows the designer to take into account steady state specifications expressed as lower limits of the classical error coefficients K (position), K (velocity), and K (acceleration).

In the area of graphical solutions some overlap of fields already mentioned occurs. For example, Numakura[13] presents a graphical solution of the Hurwitz criterion in which the stable region of the system is drawn on a plane with two of the system parameters as coordinates. Further overlap between the areas of graphical solutions and compensation occurs when one realizes that the work presented by Mitrovic and Hsu involves a large amount of graphical interpretation.

While root location is important in designing a new control system, perhaps equally as important is the ability to verify pole locations in existing systems. Lendaris, Smith[14], and Brussolo[15] have devised a method of doing this. In their method, use is made of a "zero generator." The signal from this generator is fed into the system to cancel existing poles. A null method is used to determine when the pole is cancelled, and the position of the zero is given by the calibration of the zero generator. Brussolo's method is an extension of the Lendaris, Smith method applied to more complicated systems.

No discussion of work done on control system design based on root location would be complete without mention of Truxal[16], Gibson[17], Brown[18], and Savant[19]. These men have been the link between the old and the new methods of control system design. Since Evans presented his "Graphical Analysis of Control Systems" in 1948, these men have amplified, developed, and made practical his work and that of others in this field. The textbooks, reference material, and papers they have done have enlightened the engineering world as to the precepts and possibilities of the field of automatic control system design.

CHAPTER III

PROCEDURE

To begin an investigation of the effects of root location on control system design, the author first felt it necessary to establish graphically how a high-order system may be made equivalent to a chosen second order system. As stated in the previous chapter this can be done by pushing the extra roots of the high order system far out the negative real axis so that their effect on the system's transient response is negligible. Then the two dominant roots essentially describe the system response. A standard second order system with a damping ratio of 0.707 and a natural frequency of 1.414 was chosen to illustrate this procedure.

Substituting these values into the standard form of the characteristic equation of a second order system which is

$$s^2 + 2\,\zeta\omega_n s + \omega_n^2 = 0 \tag{6}$$

where

$$\zeta = 0.707$$

$$\omega_n = 1.414$$

gives

$$s^2 + 2s + 2 = 0 \qquad (7)$$

The roots of equation (7) are $-1 \pm j$. In order for a higher order system to have a response that is approximated by that of a second order system with the above characteristic equation, it must have a pair of dominant roots of this same value. The extra roots of the system must be placed so that their real parts are further out the real axis than the point (-1,0). Figures 3, 4, and 5 show the transient response of third, fourth, and fifth order systems, respectively, to a step input for various root positions found using an Ease analog computer.

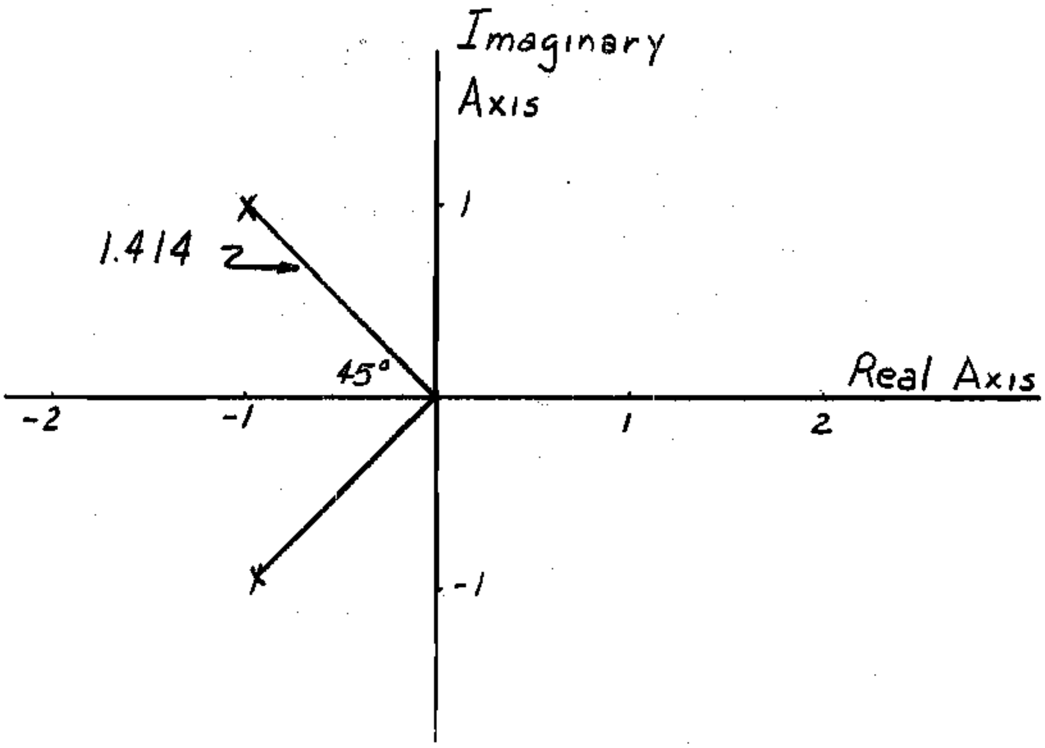


Figure 2. s Plane Plot of Second Order System

In the third order system the one extra root was positioned at first one, then two, three, five, seven, and finally ten times the real part of the chosen, dominant, complex pair. Figure 3 depicts the results of this process.

In the fourth order case the two extra roots were selected in all instances save one as a complex pair with real part first two, then three, four, five, seven, ten and finally 20 times the real part of the chosen, dominant, complex pair. In the one instance where this was not done, the two extra roots were chosen as real roots whose average magnitude was three times the real part of the chosen dominant complex pair. Figure 4 indicates how the fourth order system compares to the chosen second order system.

In the fifth order case, the three extra roots were chosen in all but one case as a complex pair plus one real root equal in magnitude to the real part of that complex pair. These three roots were stepped out the negative real axis with their real part magnitudes equal to first two, then three, four, five, ten, and finally 20 times the real part of the chosen, dominant, complex pair. The curves in Figure 5 are the results of this process. In one case all three extra roots were chosen real with average magnitude equal to three times the real part of the chosen dominant complex pair. This illustrates the difference in response for one system having three extra real roots, and another system having one extra real root and an extra complex pair where the average real part magnitude of both systems is equal.

In the fourth and fifth order cases where the extra roots were assumed as complex pairs, the positions of these pairs were changed

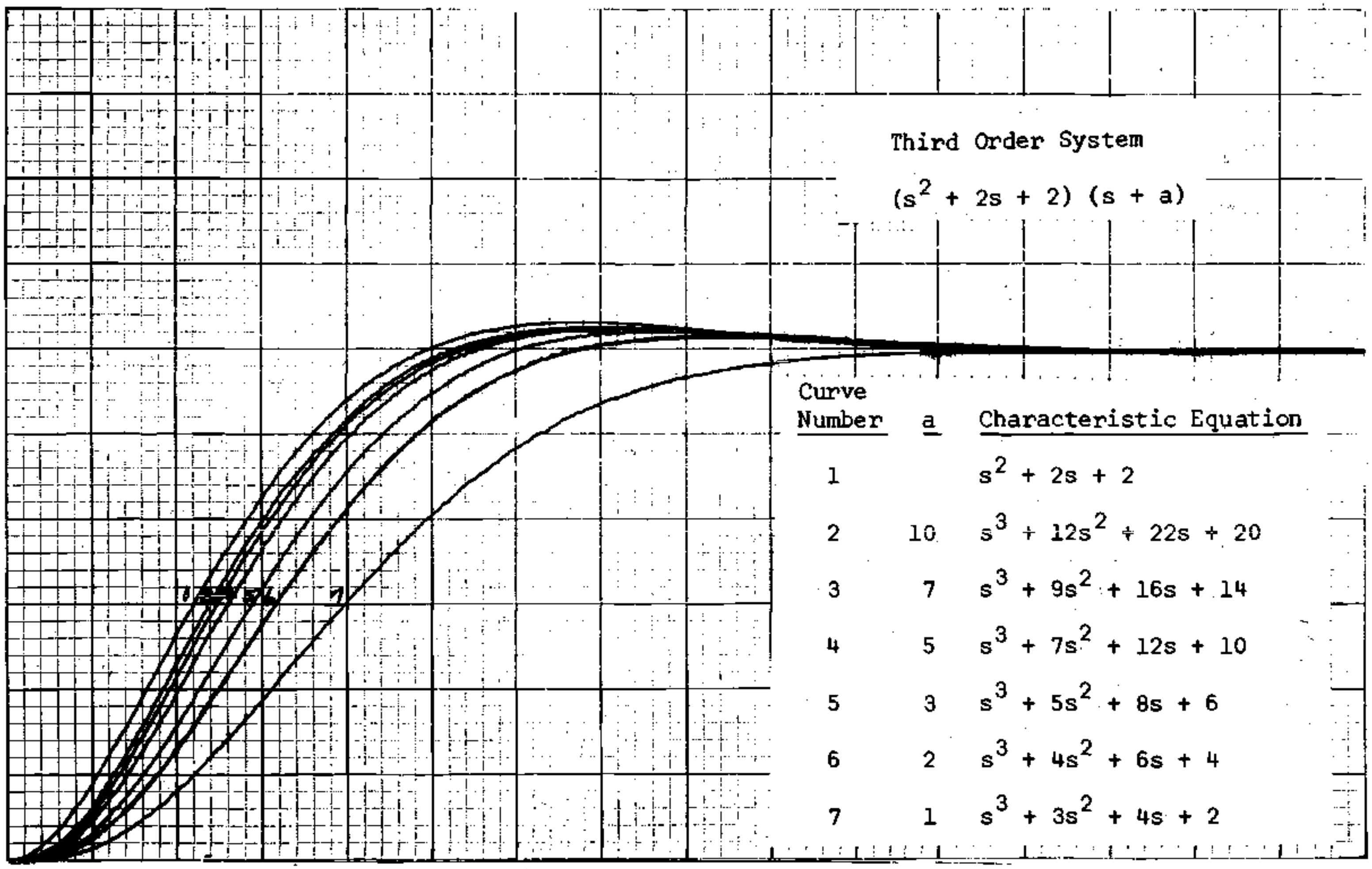
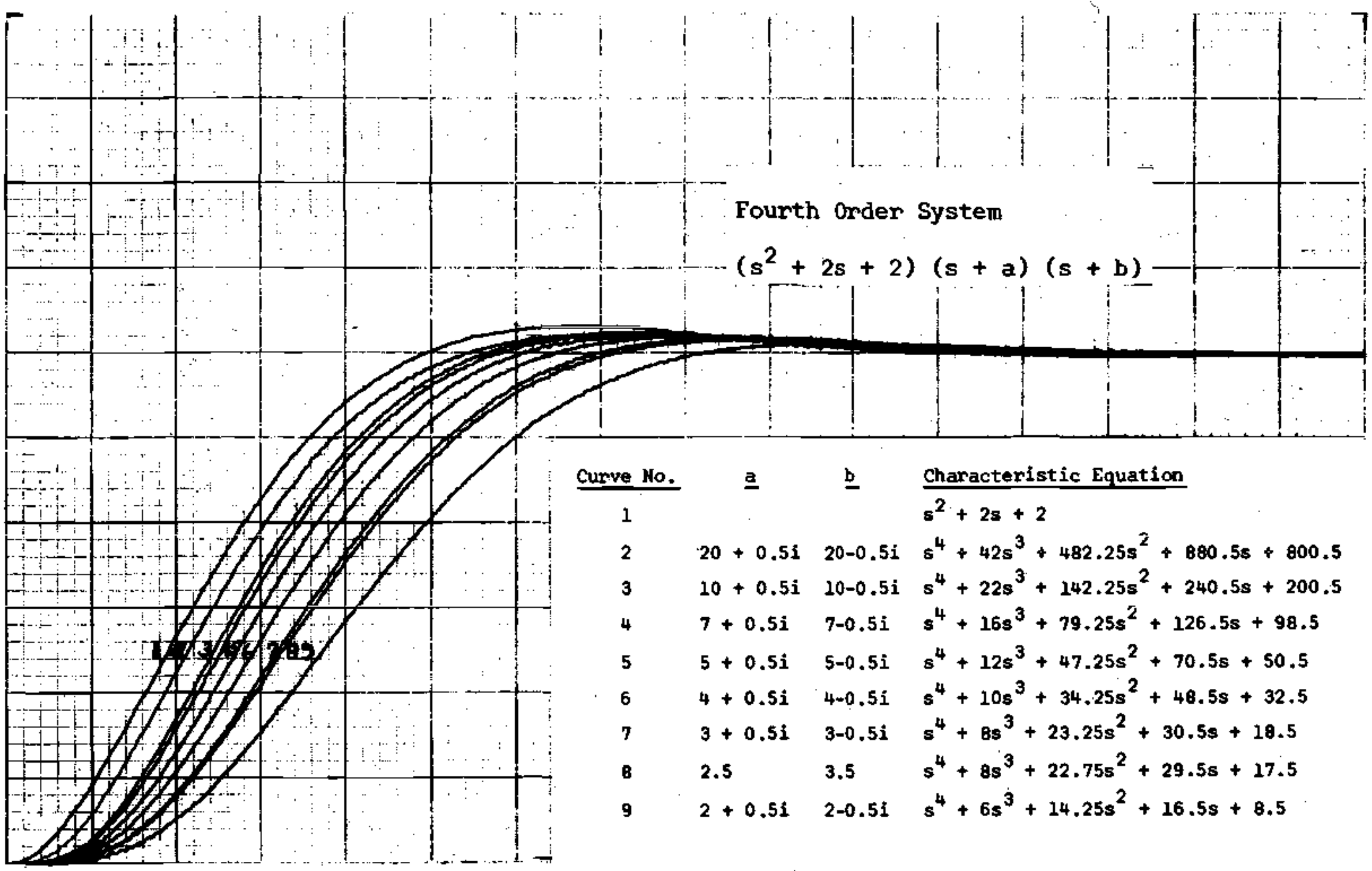


Figure 3. Second Order Approximation of a Third Order System



Curve No.	a	b	Characteristic Equation
1			$s^2 + 2s + 2$
2	20 + 0.5i	20-0.5i	$s^4 + 42s^3 + 482.25s^2 + 880.5s + 800.5$
3	10 + 0.5i	10-0.5i	$s^4 + 22s^3 + 142.25s^2 + 240.5s + 200.5$
4	7 + 0.5i	7-0.5i	$s^4 + 16s^3 + 79.25s^2 + 126.5s + 98.5$
5	5 + 0.5i	5-0.5i	$s^4 + 12s^3 + 47.25s^2 + 70.5s + 50.5$
6	4 + 0.5i	4-0.5i	$s^4 + 10s^3 + 34.25s^2 + 48.5s + 32.5$
7	3 + 0.5i	3-0.5i	$s^4 + 8s^3 + 23.25s^2 + 30.5s + 18.5$
8	2.5	3.5	$s^4 + 8s^3 + 22.75s^2 + 29.5s + 17.5$
9	2 + 0.5i	2-0.5i	$s^4 + 6s^3 + 14.25s^2 + 16.5s + 8.5$

Figure 4. Second Order Approximation of a Fourth Order System

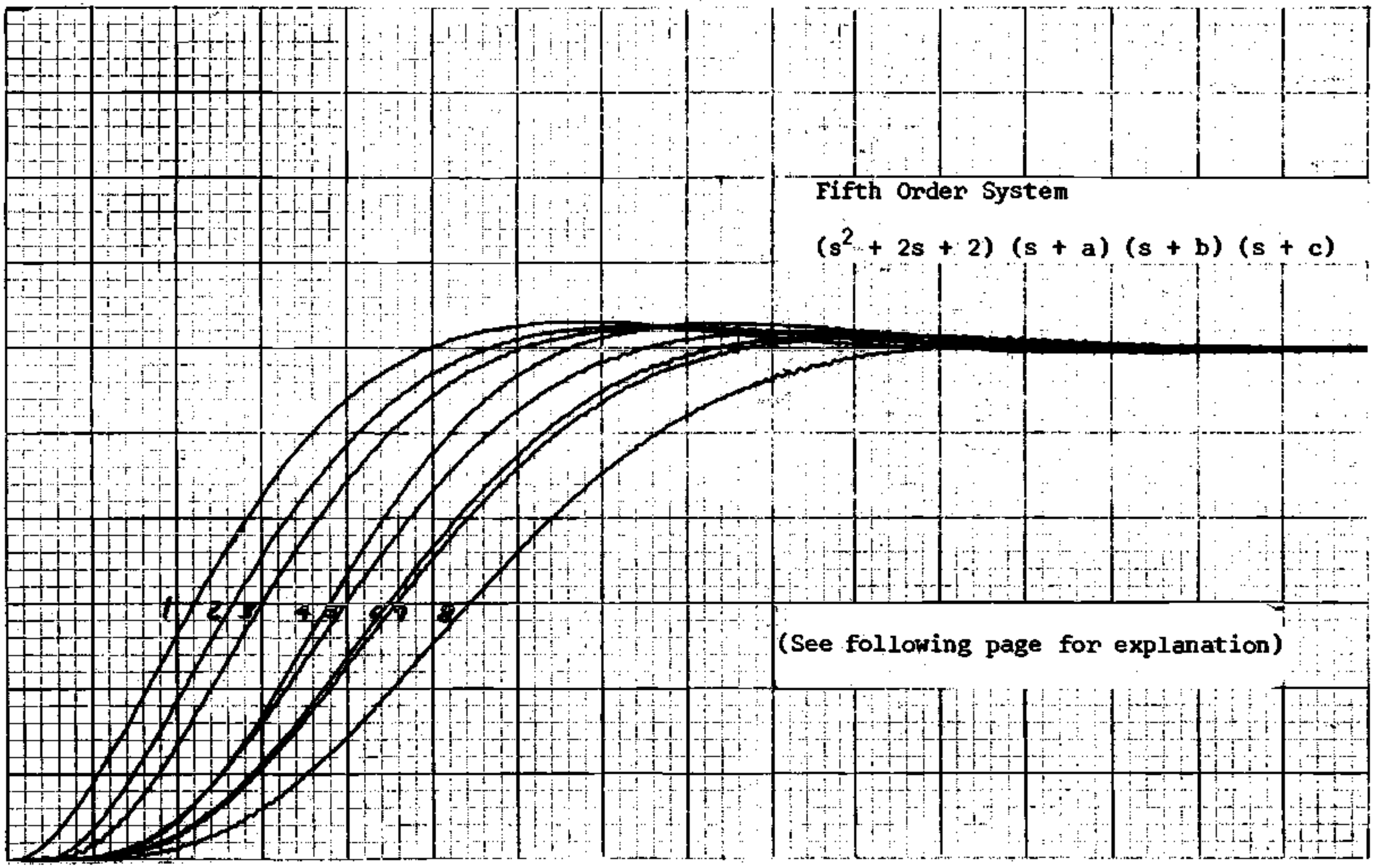


Figure 5. Second Order Approximation of a Fifth Order System

Figure 5

Curve Number	a	b	c	Characteristic Equation
1				$s^2 + 2s + 2$
2	20+0.5i	20-0.5i	20	$s^5 + 62s^4 + 1322.0s^3 + 10{,}520s^2 + 18{,}400s + 16{,}000$
3	10+0.5i	10-0.5i	10	$s^5 + 32s^4 + 362.25s^3 + 1663.0s^2 + 2605.5s + 2005.0$
4	5+0.5i	5-0.5i	5	$s^5 + 17s^4 + 107.25s^3 + 306.75s^2 + 403.0s + 252.5$
5	4+0.5i	4-0.5i	4	$s^5 + 14s^4 + 74.25s^3 + 185.25s^2 + 226.5s + 130$
6	3+0.5i	3-0.5i	3	$s^5 + 11s^4 + 47.25s^3 + 100.25s^2 + 110.0s + 55.5$
7	2.5	3.5	3	$s^5 + 11s^4 + 46.75s^3 + 98s^2 + 106.0s + 52.5$
8	2+0.5i	2-0.5i	2	$s^5 + 8s^4 + 26.25s^3 + 45s^2 + 41.5s + 17.0$

by varying the real part only. Since this real part furnishes the exponent of the exponential term, it determines the time for the effect of that root to decay to zero. The results obtained (Figures 3, 4, and 5) are curves in which the extra complex pairs have imaginary parts equal to that of the dominant pair. This was an arbitrary choice made on the basis that little change in system response was noted when the extra roots where chosen as a complex pair or as real roots with real part values nearly equal to that of the complex pair. Curves numbered seven and eight of Figure 4 and six and seven of Figure 5 illustrate this characteristic.

Figure 1 in Chapter 1 indicates a basic feedback control system. To achieve a reasonably short settling time with high order feedforward transfer functions (G(s)), it is often necessary to feed back not only the controlled variable (C(s)), but its derivatives. Feedback of the signals K_1C, $K_2\dot{C}$, and $K_3\ddot{C}$ is conventionally denoted by position, velocity, and acceleration feedback where the constants K_1, K_2, and K_3 are feedback gains. The locations of the roots of the characteristic equation are affected by these gains.

A Burroughs 220 Digital Computer was used to determine the value of gains for various root combinations. An example of the programs used may be found in Appendix B. With these programs the feedback gain constants necessary were determined for third, fourth, or fifth order systems so that their behavior approximated that of a second order system to various degrees of accuracy.

In summation of the procedure: systems up through fifth order were simulated on an analog computer. From the resulting curves information was obtained regarding how far out the negative axis in the s plane the extra roots of high order systems must be placed to simulate a chosen second order system. Then, using a digital computer, information was obtained and various curves plotted to enable a control system designer to choose a second order system and design systems up through order five that are equivalent to this chosen system. This can be done using appropriate feedback information obtained from the plotted curves.

CHAPTER IV

RESULTS

The basic results obtained in this study are shown in Figures 7 through 15.

In a normal design situation the designer may know or be restricted in choosing, feedforward transfer function coefficients, but usually he will have more latitude in deciding what K_i (feedback gains) to use. From Equation 3 (Chapter I) one can see that it is the sum of these two elements that are determined by root location, or vice versa. That is, if this sum is preselected for all coefficients, then the root locations are specified. Conversely, if the root locations are chosen first, then the characteristic equation coefficients are specified. The control engineer might normally choose his root locations first to obtain a given system performance, and with this information determine the necessary values for a, b,---n and K_i.

As an example, assume a system as shown in Figure 6. Here the characteristic equation is

$$s^3 + (K_3 + a)s^2 + (K_2 + b)s + (K_1 + c) = 0 \tag{8}$$

Assume the designer wants system performance characteristics equivalent to that of a second order system defined by a damping ratio of 0.6 and a natural frequency of 4.0 rad/sec. Using the form of Equation (6) the characteristic equation of this second order system would be

$$s^2 + 4.8s + 16 = 0 \tag{9}$$

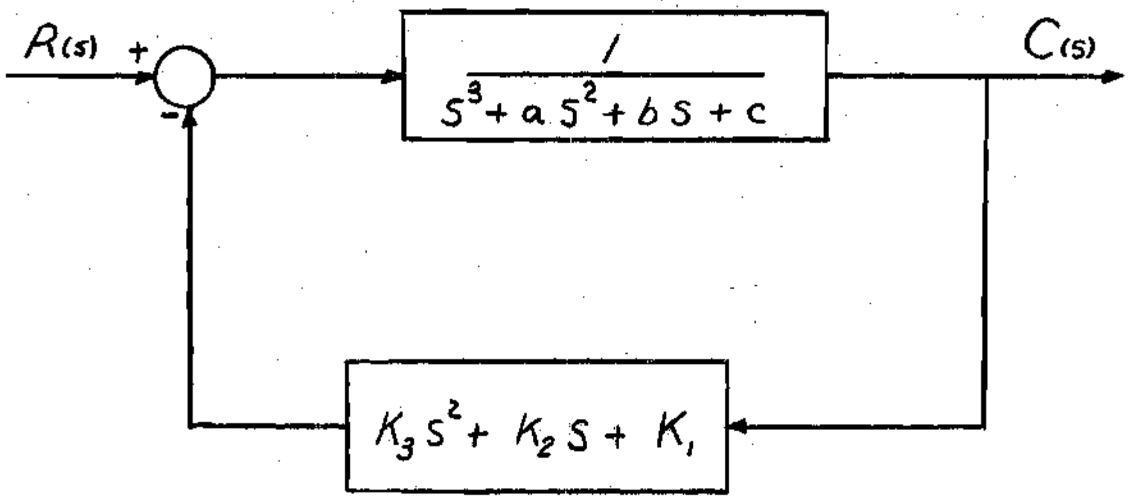


Figure 6. Example One

If the feedforward transfer function represented a motor-generator set, for example, then the constants a, b, and c would be fixed by system characteristics such as generator resistance, motor resistance, and motor and load inertia. It would then be the designer's job to determine those values of K_i to make this third order system (Equation 8) equivalent to the chosen second order system (Equation 9). That is, he must choose values for K_i such that two of the roots of the characteristic equation will be at $2.4 \pm 2\sqrt{3}\ j$ as defined by Equation 9, and the third sufficiently far out the negative real axis as to make its effect negligible.

The designer need only consult Figure 7 to determine the values $(K_1 + c) = 110$, $(K_2 + b) = 50$, and $(K_3 + a) = 12$. These values are the coefficients of the characteristic equation that has roots such that it will simulate the response of the second order system of Equation 9 where $\zeta = 0.6$ and $\omega_n = 4.0$. To obtain his feedback gains K_1, K_2, and K_3, the designer must subtract his system constants a, b, and c from those values found in Figure 7. For a more accurate approximation, Figure 8 or Figure 9 might have been used.

Preferably only position and velocity feedback would be required in the feedback element. This simplifies the system by eliminating the need to measure the second (acceleration) or higher derivatives of the output. Practically speaking, a designer would not be able to measure accurately, if at all, derivatives of the output beyond the second derivative. It is therefore almost imperative that he require no higher derivative than acceleration. In other words, for higher order systems, the K_4 and/or K_4 and K_5 constants obtained from Figures 10 through 15 would in fact have to be inherent to the given feedforward transfer function.

The curves for the higher order systems plotted in Figure 10 through 15 give gain information necessary for obtaining equivalent second order systems defined by natural frequencies from 2.0 to 20.0 rad/sec with damping ratios of from 0.4 to 1.0. These figures show plots of $(K_i$ + a constant) versus natural frequency for third, fourth, and fifth order systems, each with three degrees of equivalency. The

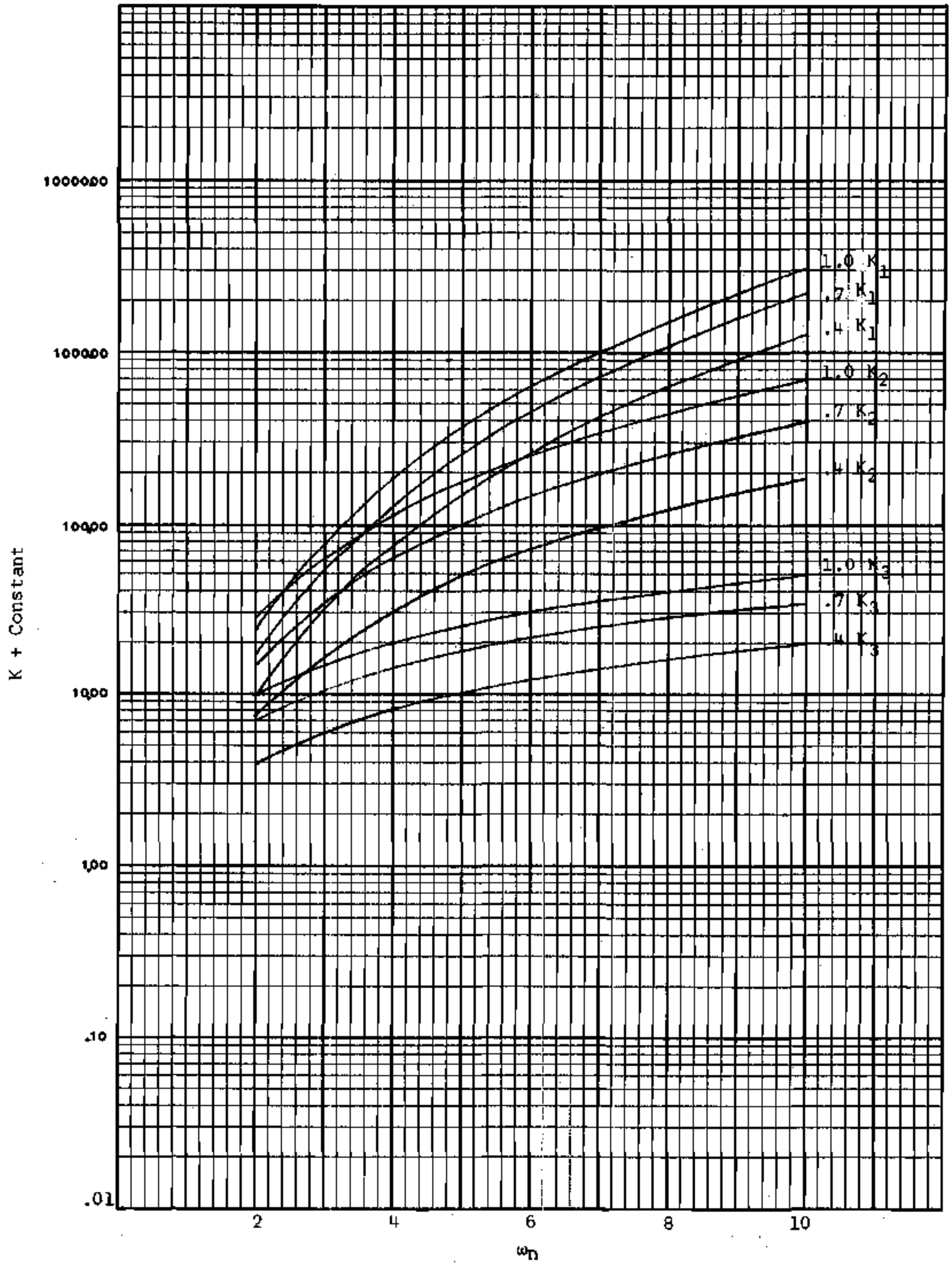


Figure 7a. Third Order System
Extra Root at Three Times Real Part

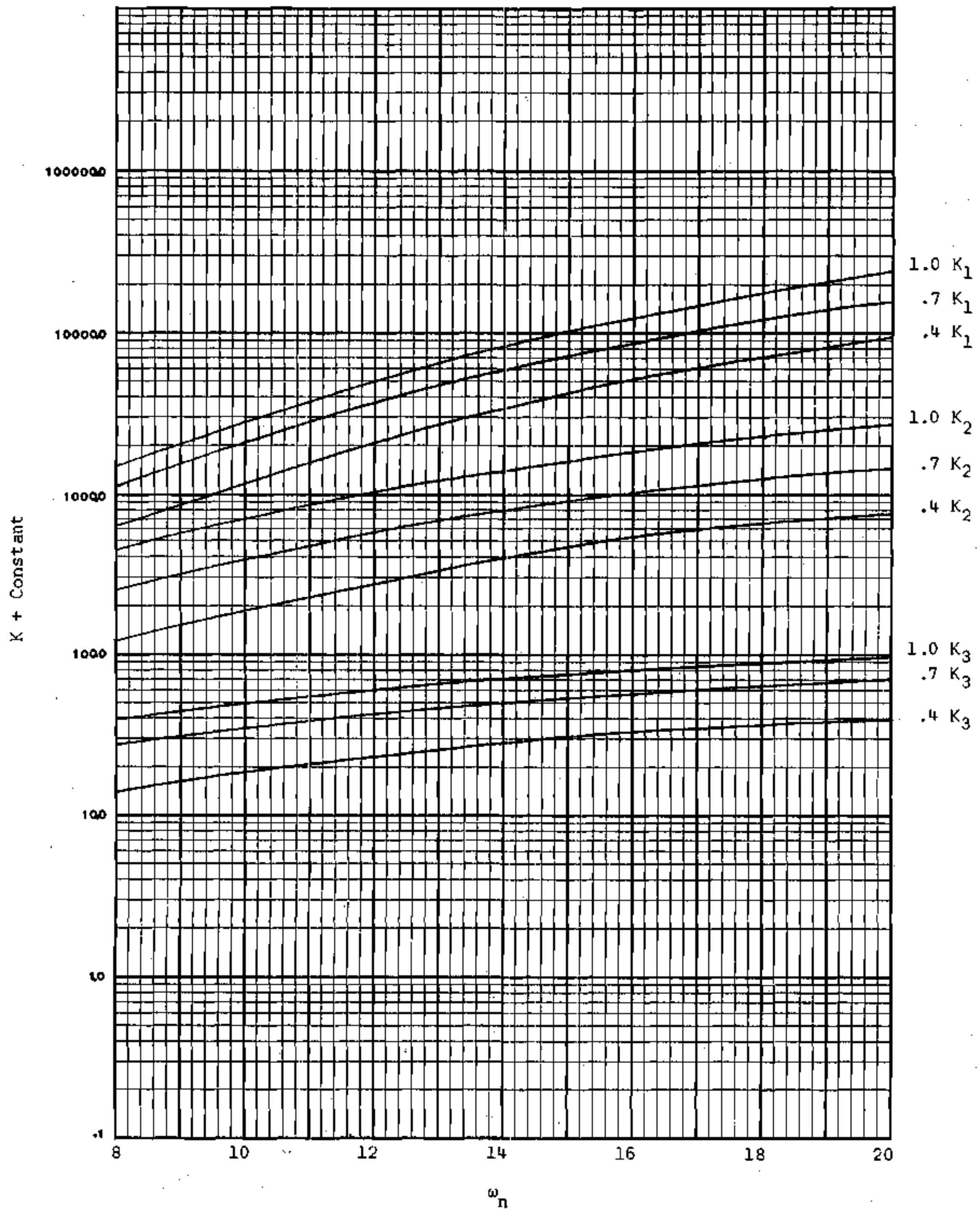


Figure 7b. Third Order System
Extra Root at Three Times Real Part

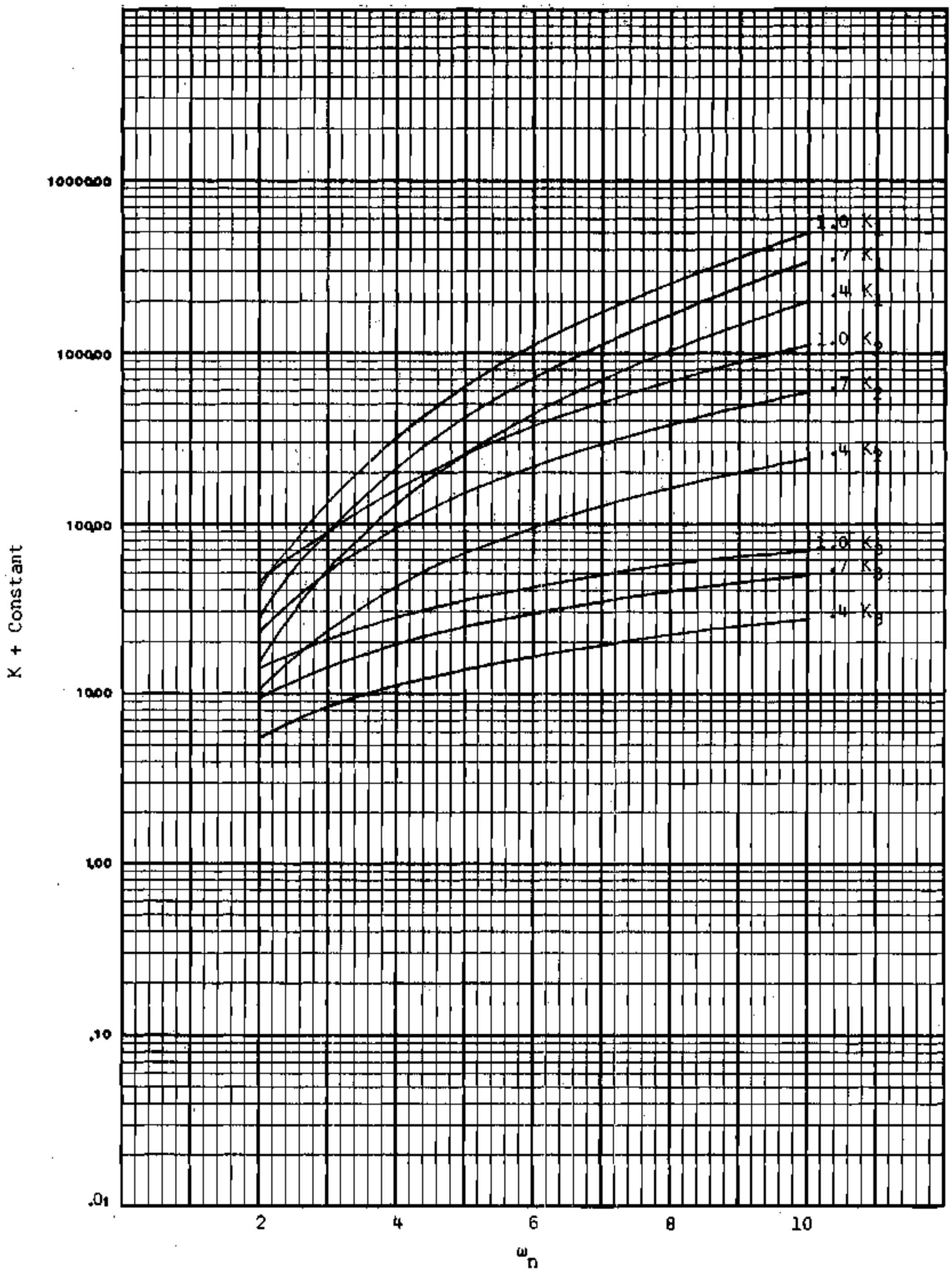


Figure 8a. Third Order System
Extra Root at Five Times Real Part

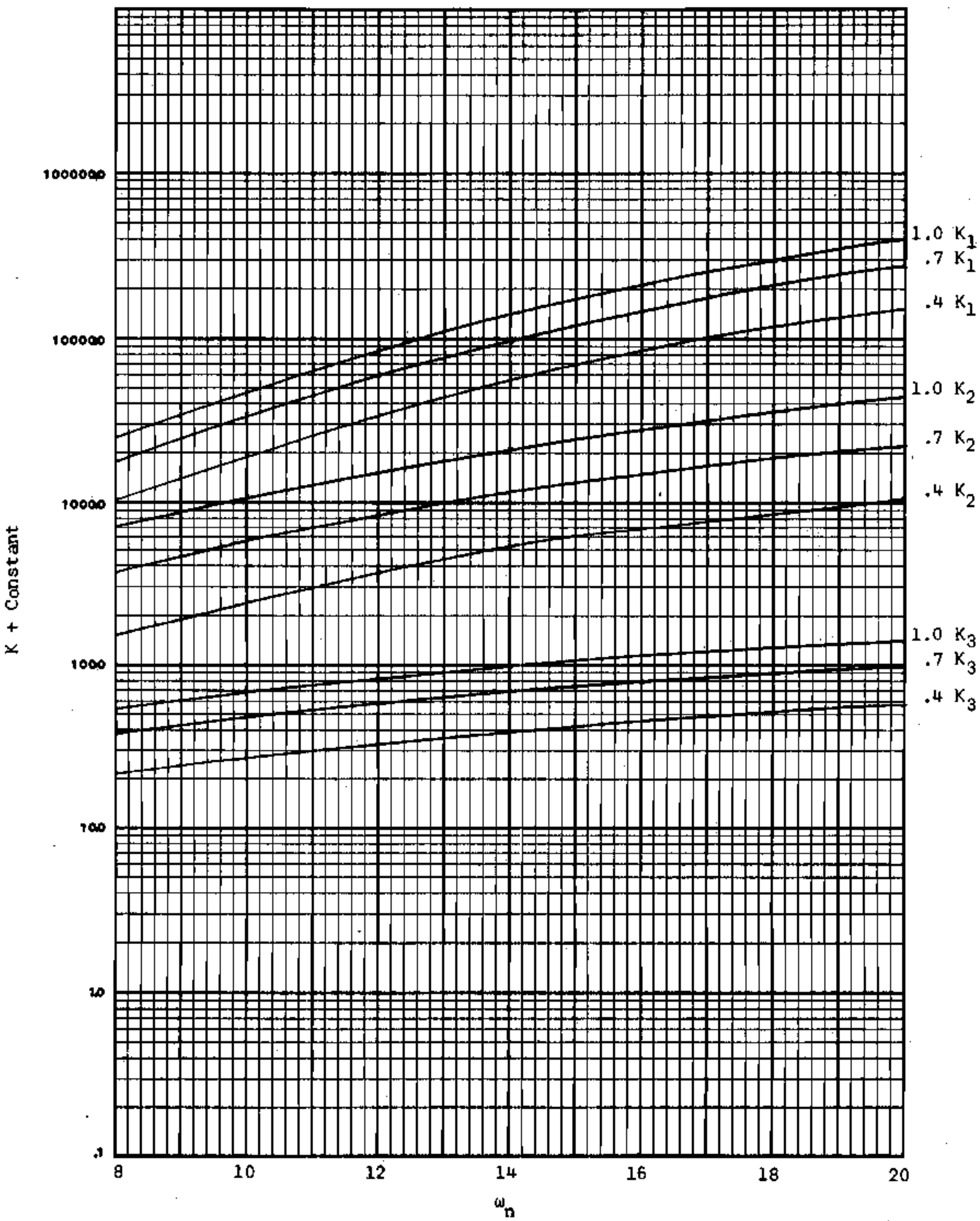


Figure 8b. Third Order System
Extra Root at Five Times Real Part

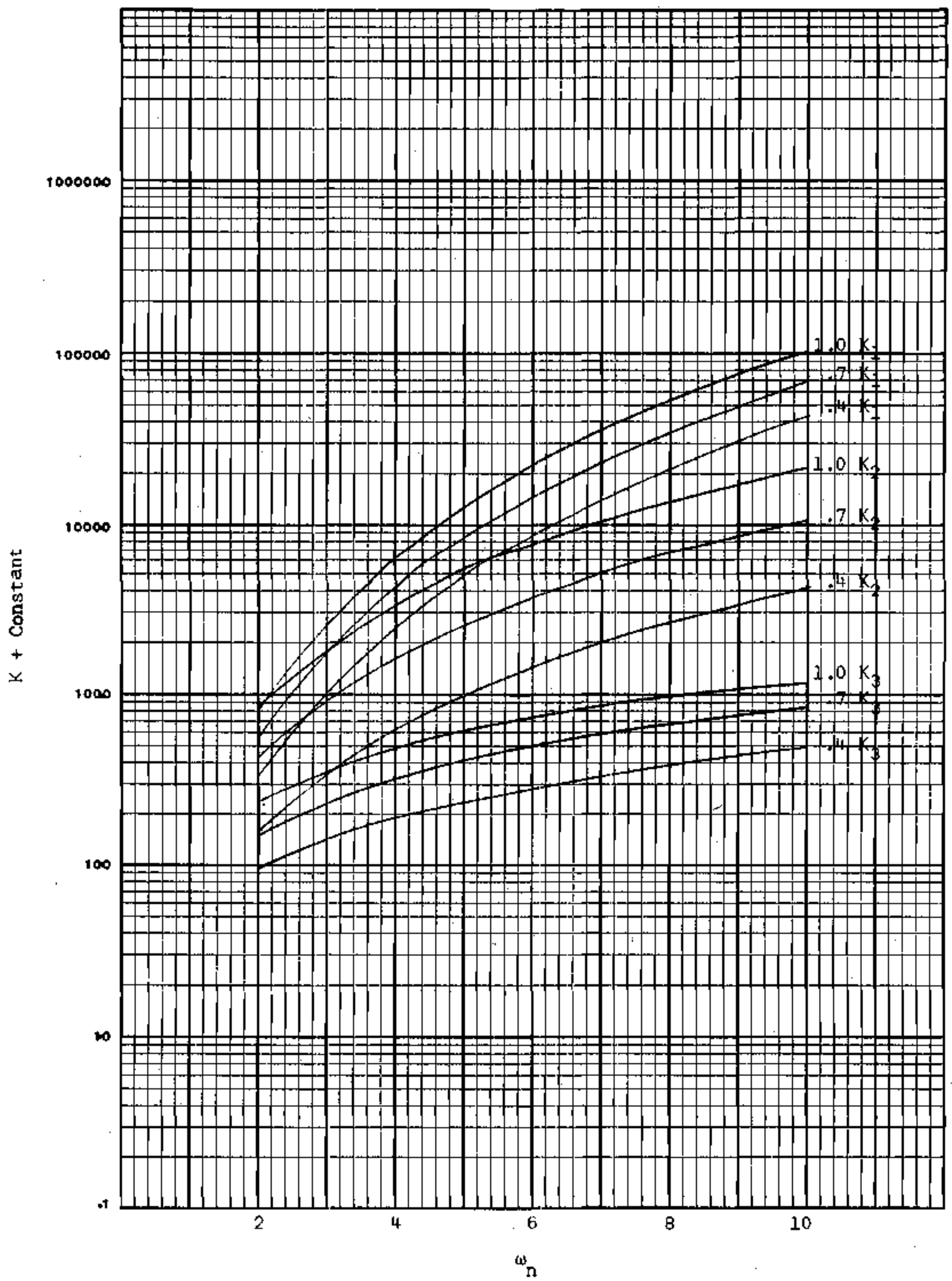


Figure 9a. Third Order System
Extra Root at Ten Times Real Part

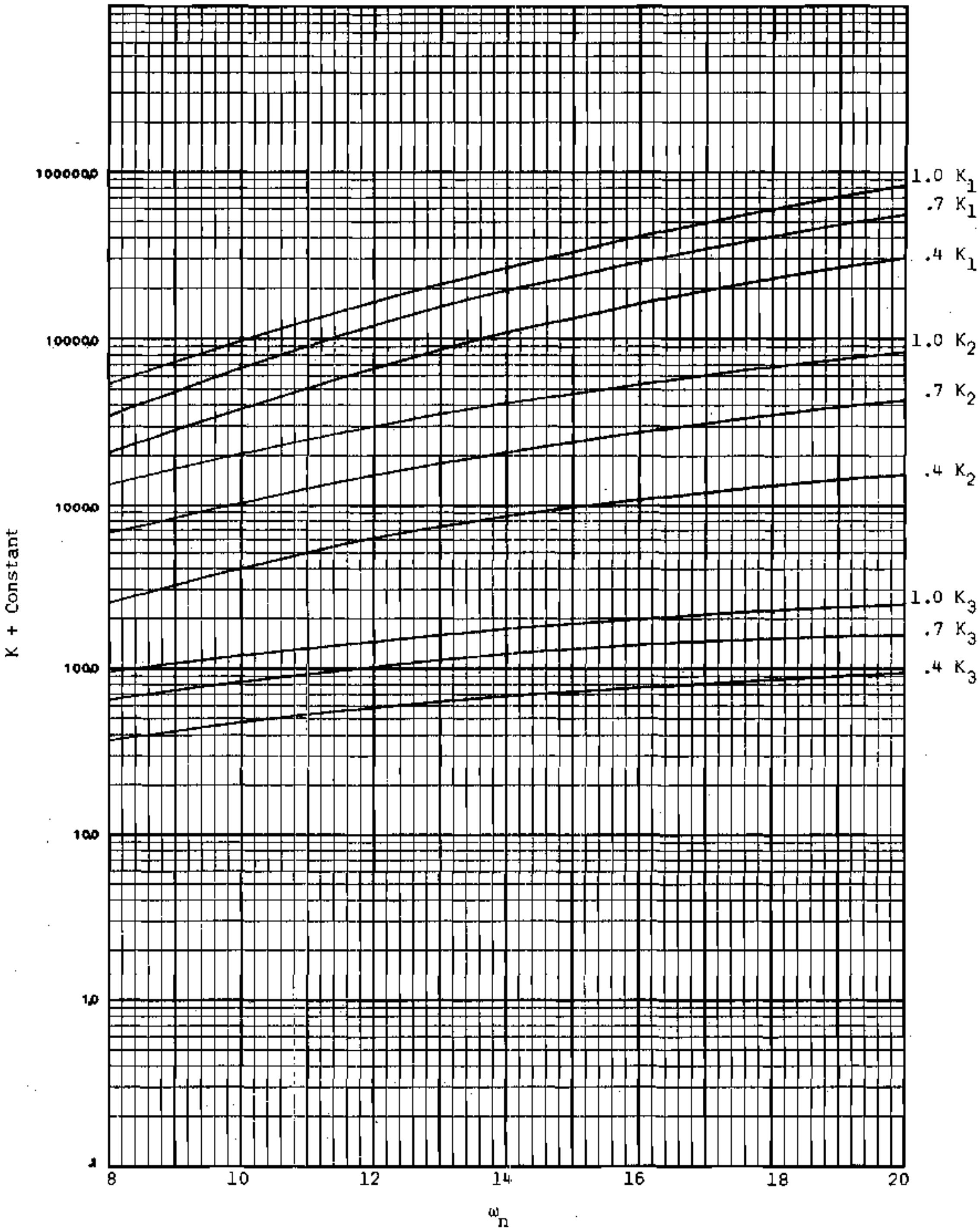


Figure 9b. Third Order System
Extra Root at Ten Times Real Part

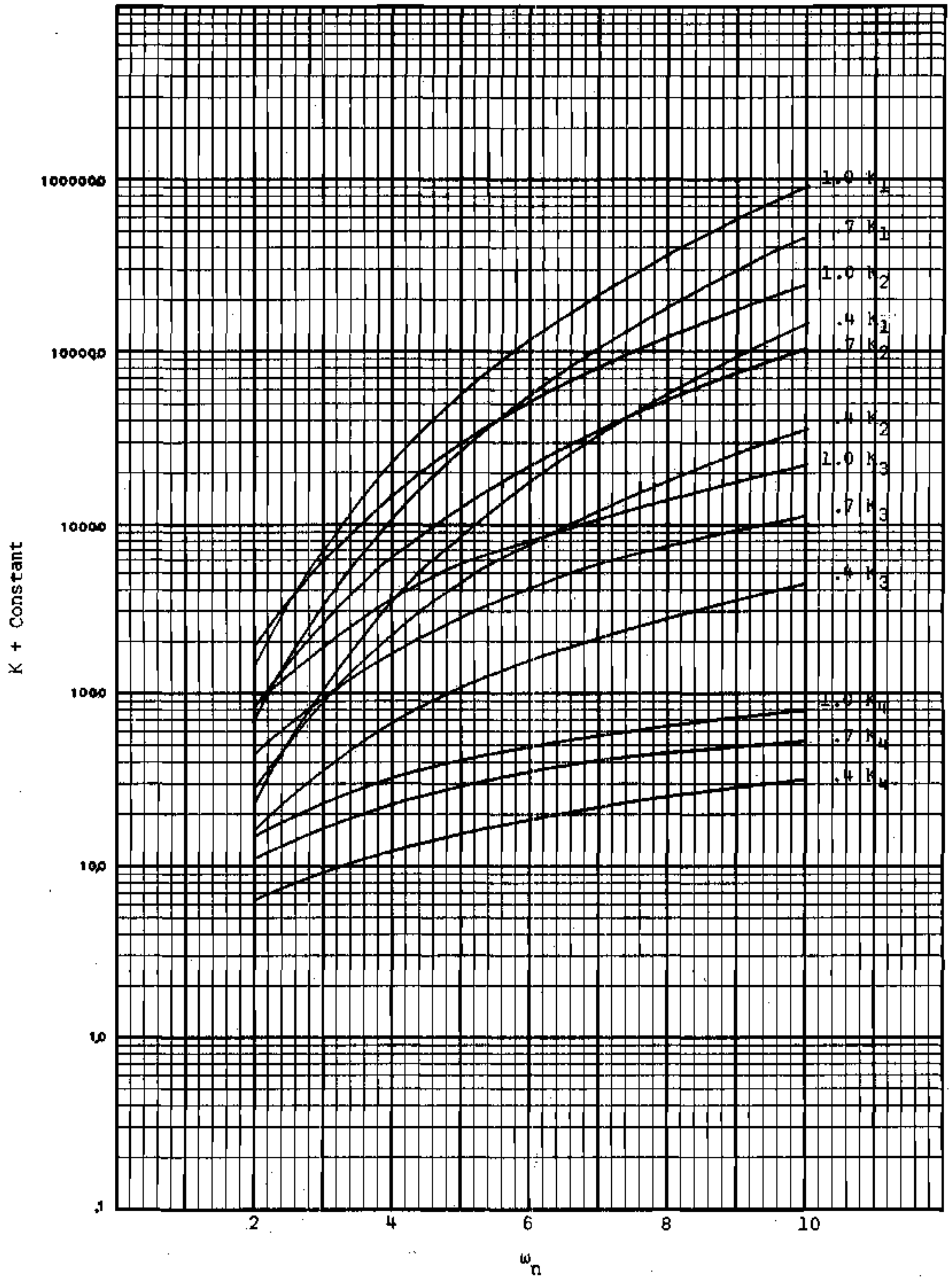


Figure 10a. Fourth Order System
Extra Roots at Three Times Real Part

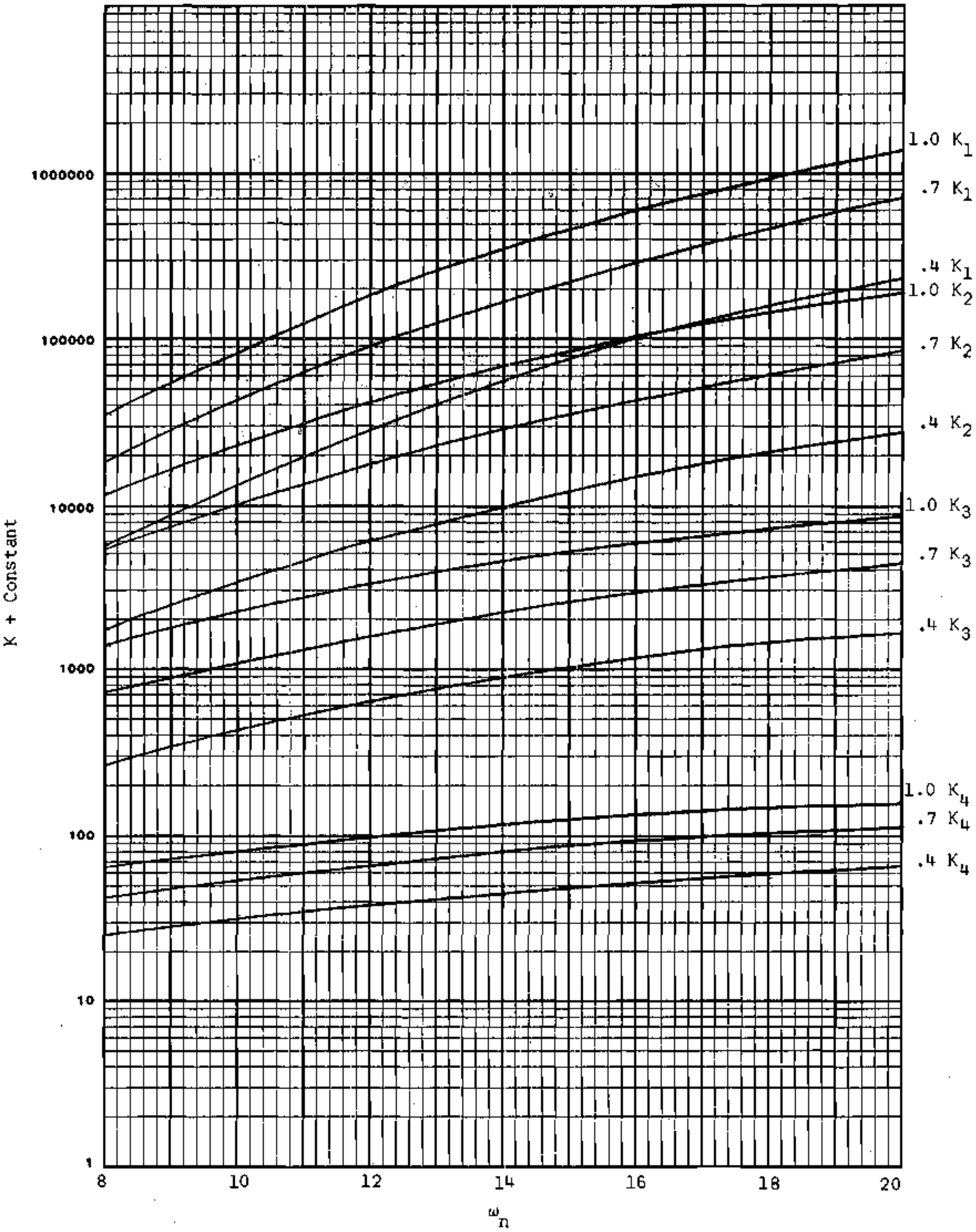


Figure 10b. Fourth Order System
Extra Roots at Three Times Real Part

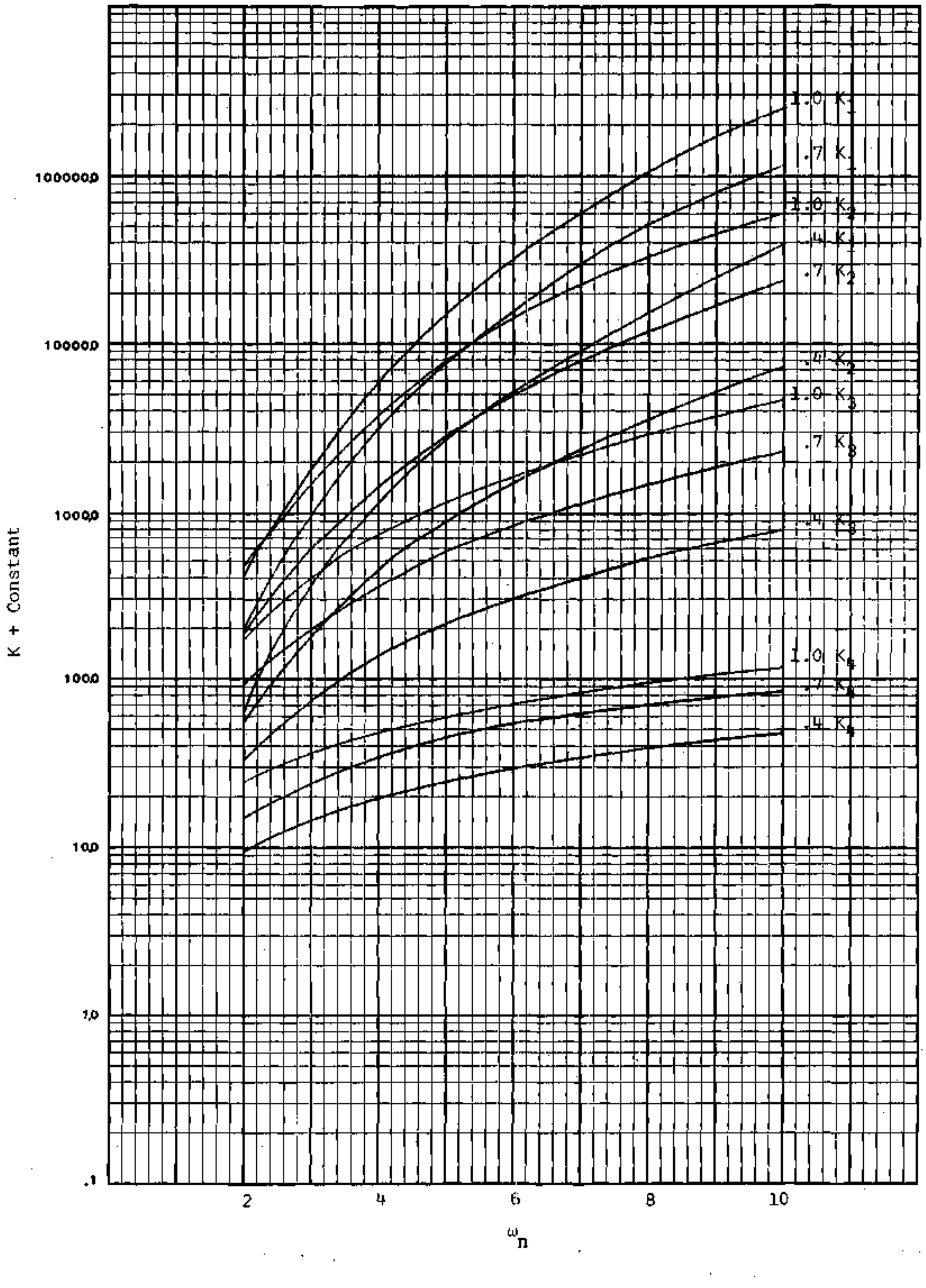


Figure 11a. Fourth Order System
Extra Roots at Five Times Real Part

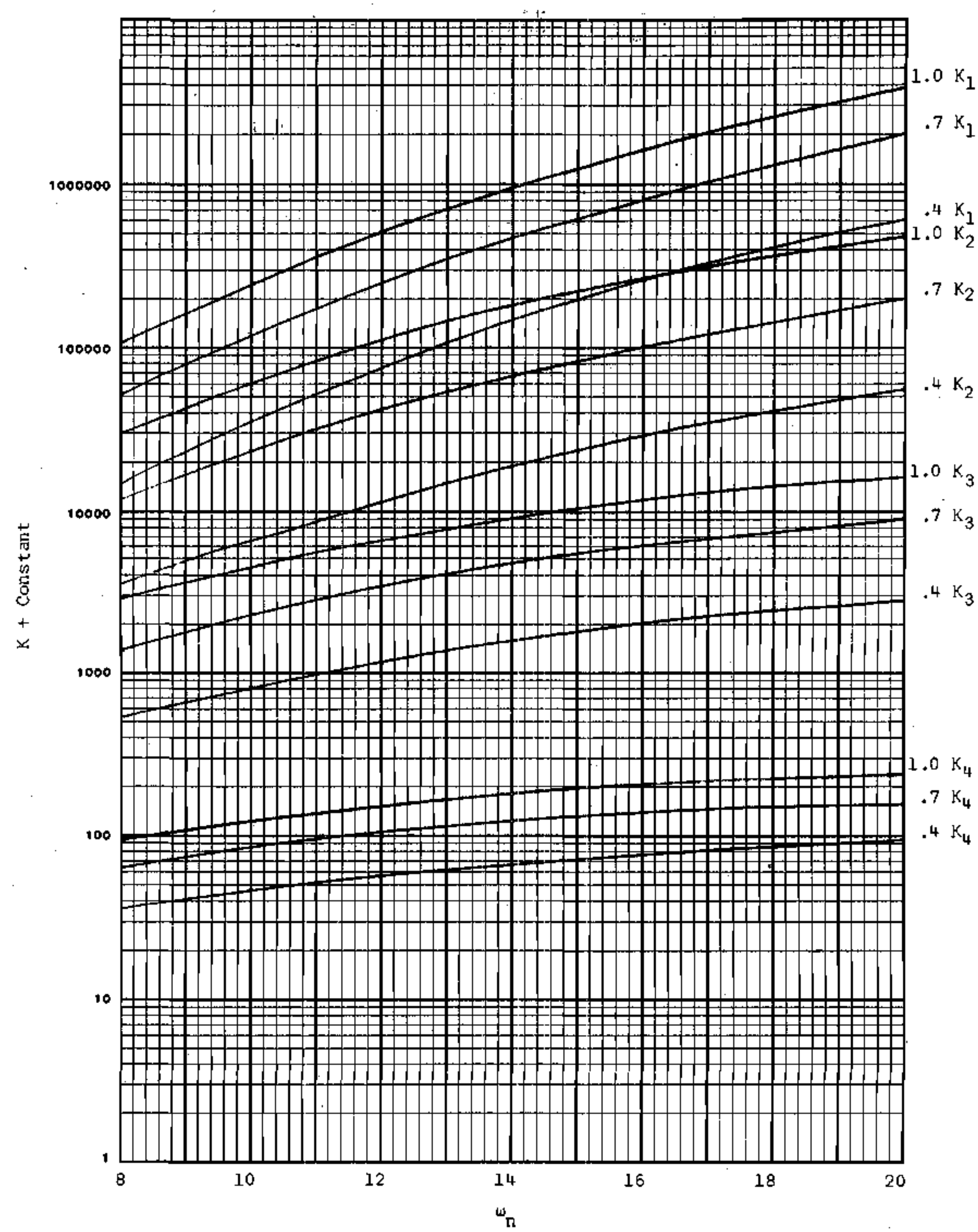


Figure 11b. Fourth Order System
Extra Roots at Five Times Real Part

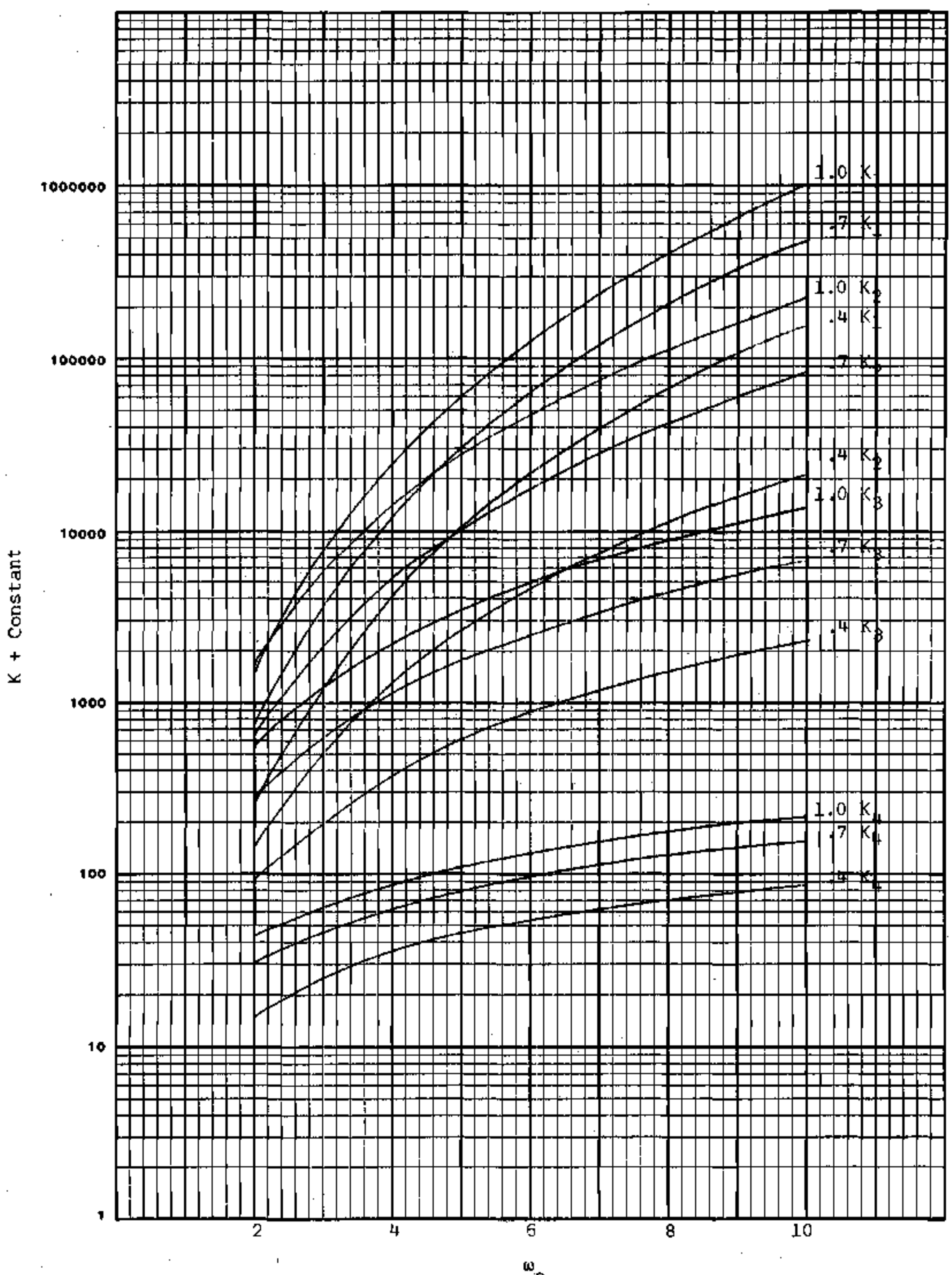


Figure 12a. Fourth Order System
Extra Roots at Ten Times Real Part

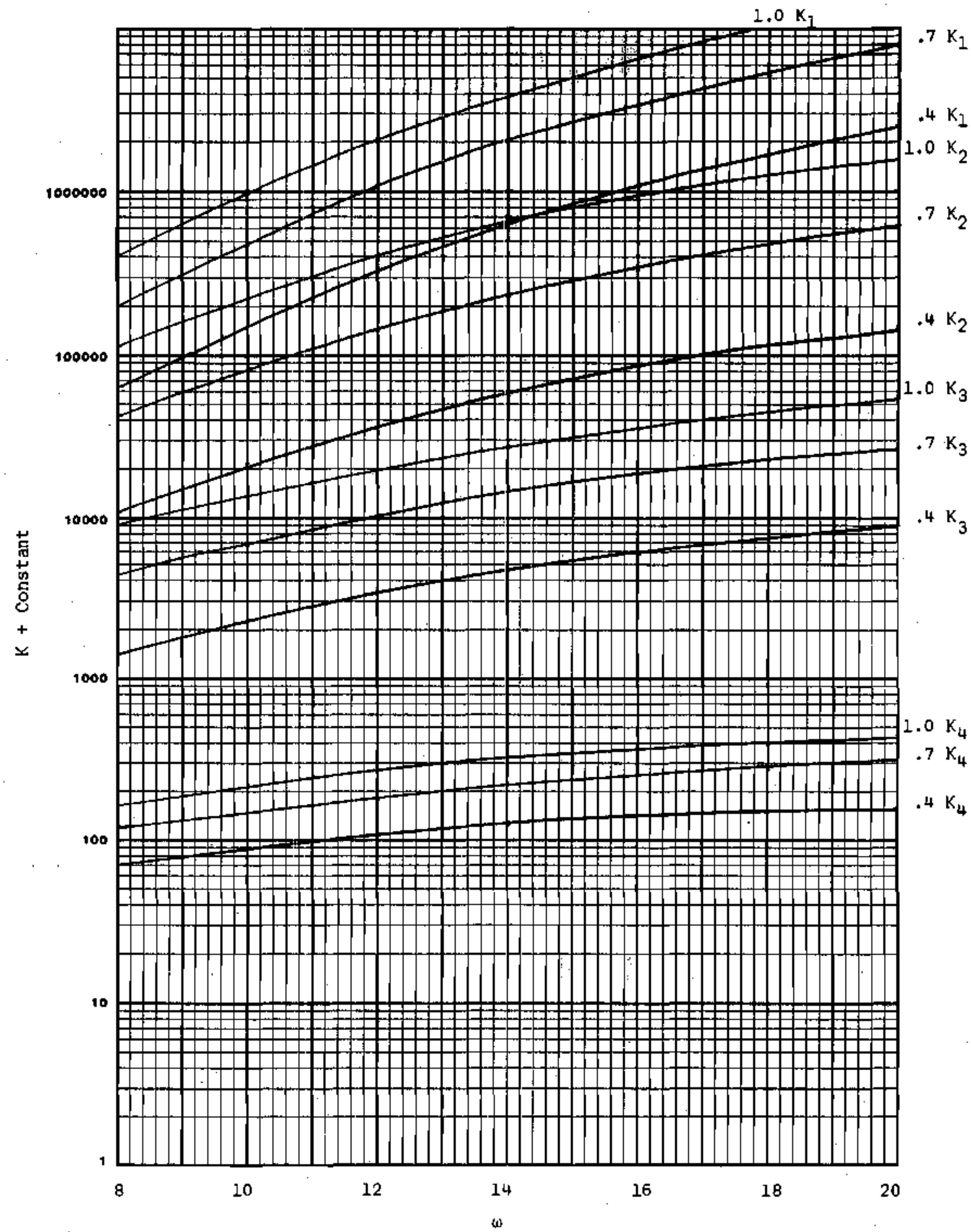


Figure 12b. Fourth Order System
Extra Roots at Ten Times Real Part

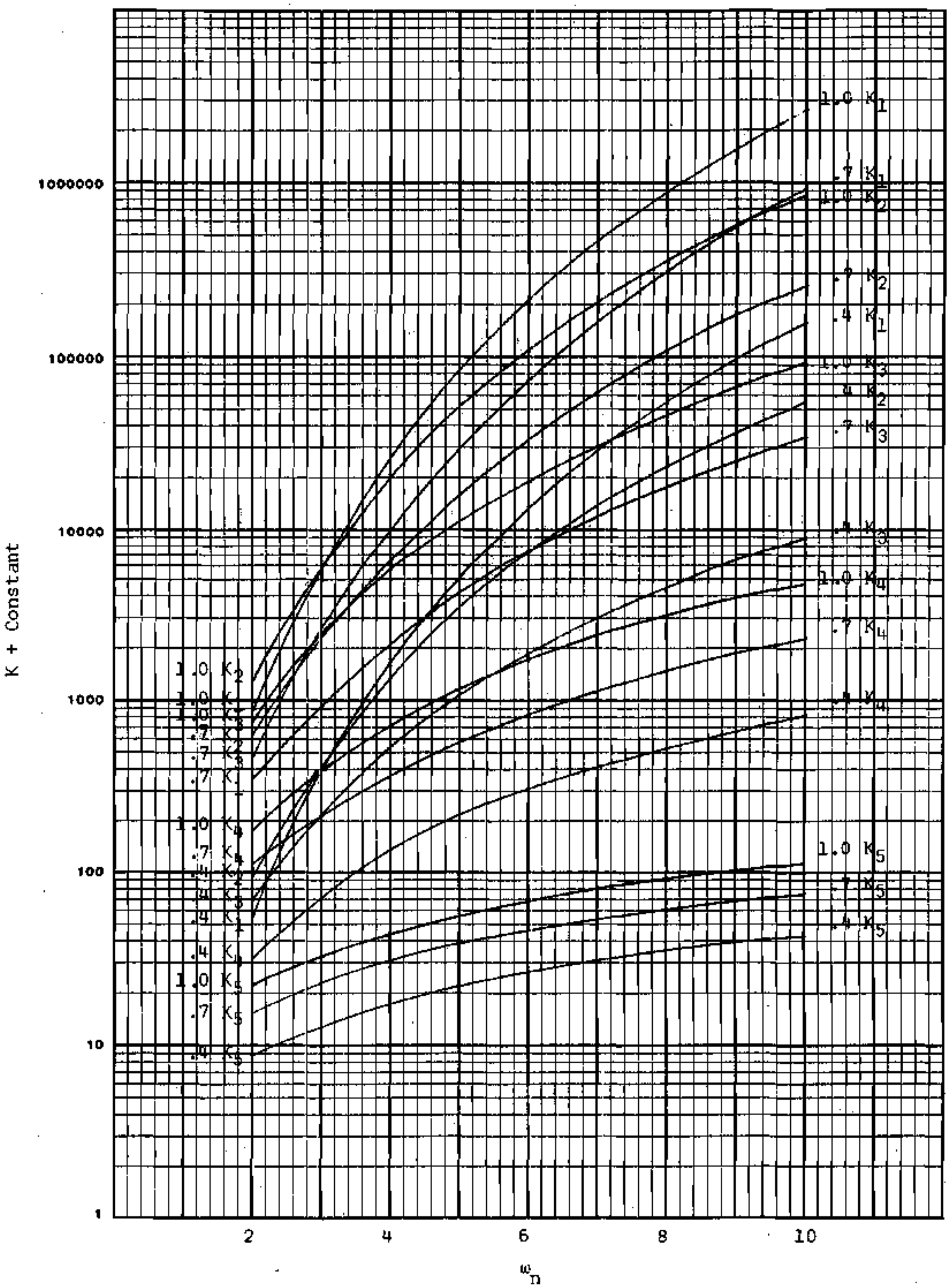


Figure 13a. Fifth Order System
Extra Roots at Three Times Real Part

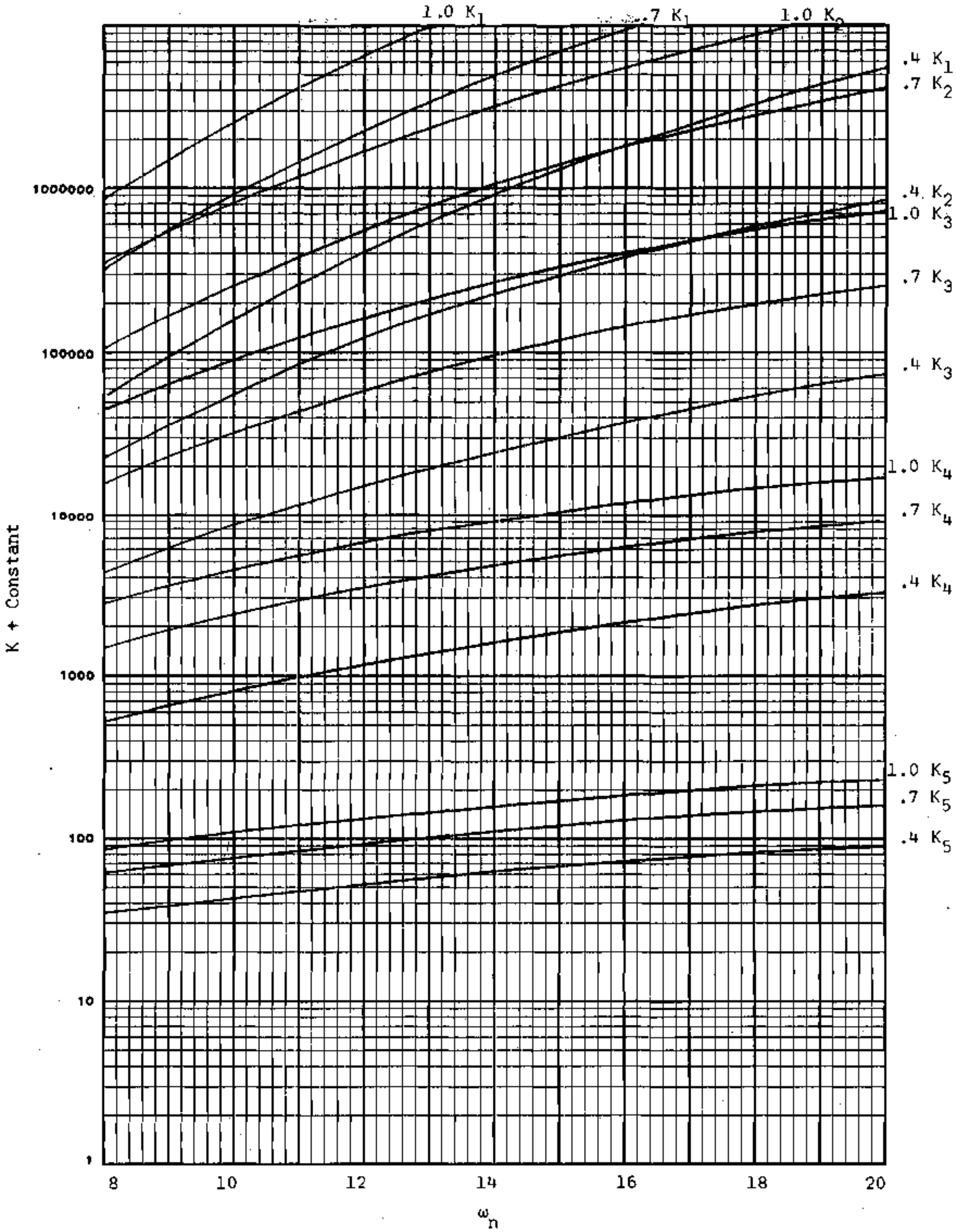


Figure 13b. Fifth Order System
Extra Roots at Three Times Real Part

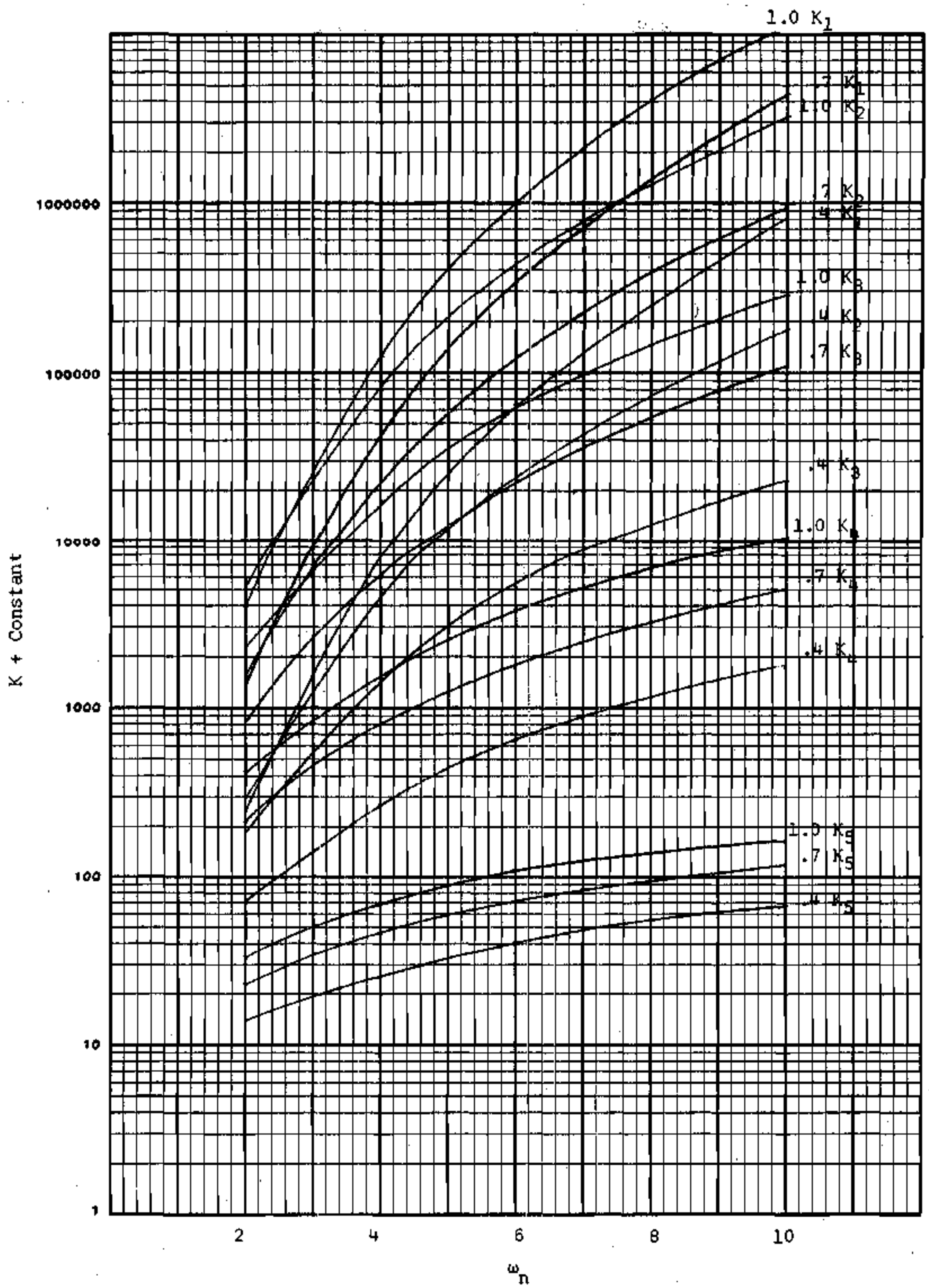


Figure 14a. Fifth Order System
Extra Roots at Five Times Real Part

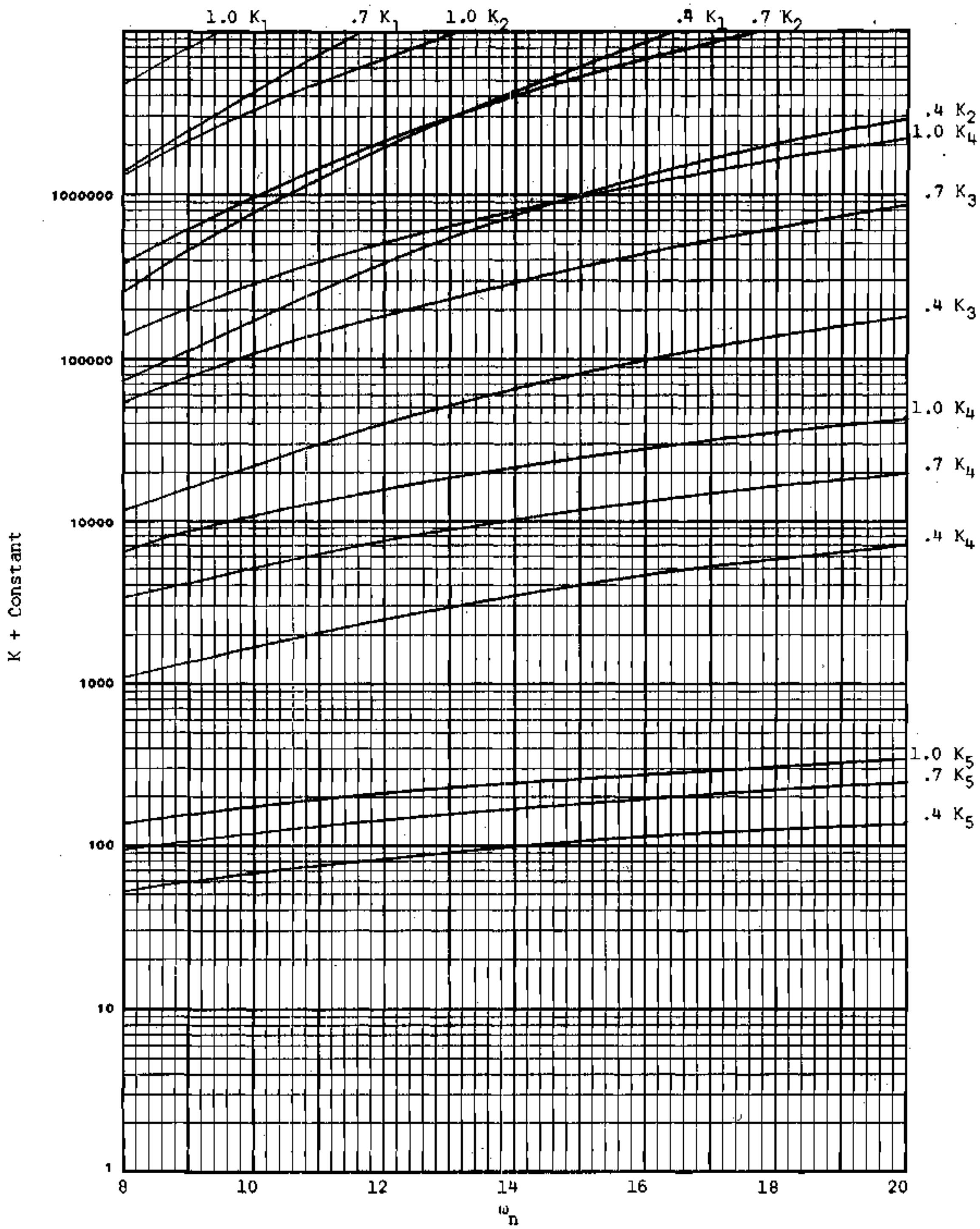


Figure 14b. Fifth Order System
Extra Roots at Five Times Real Part

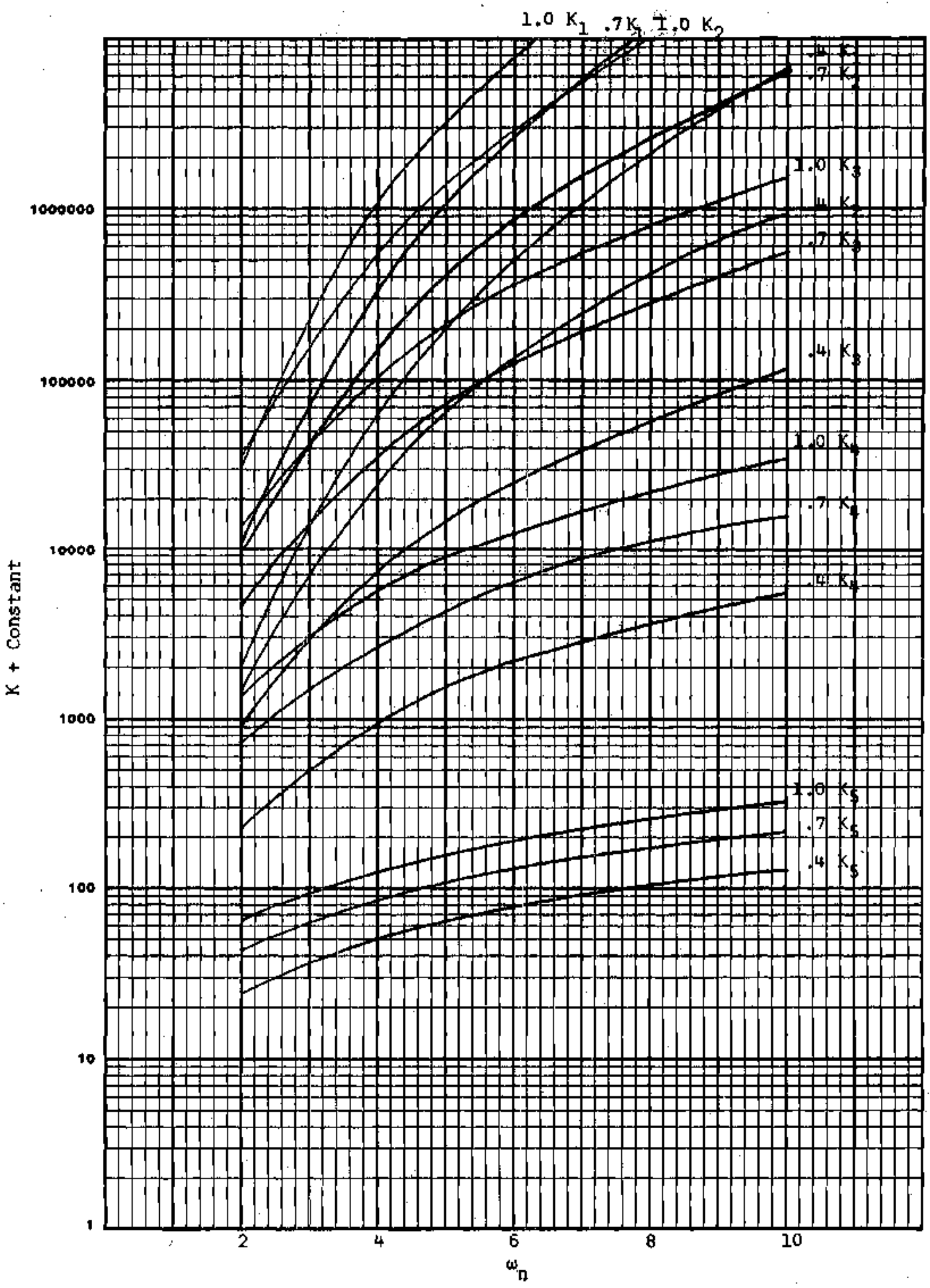


Figure 15a. Fifth Order System
Extra Roots at Ten Times Real Part

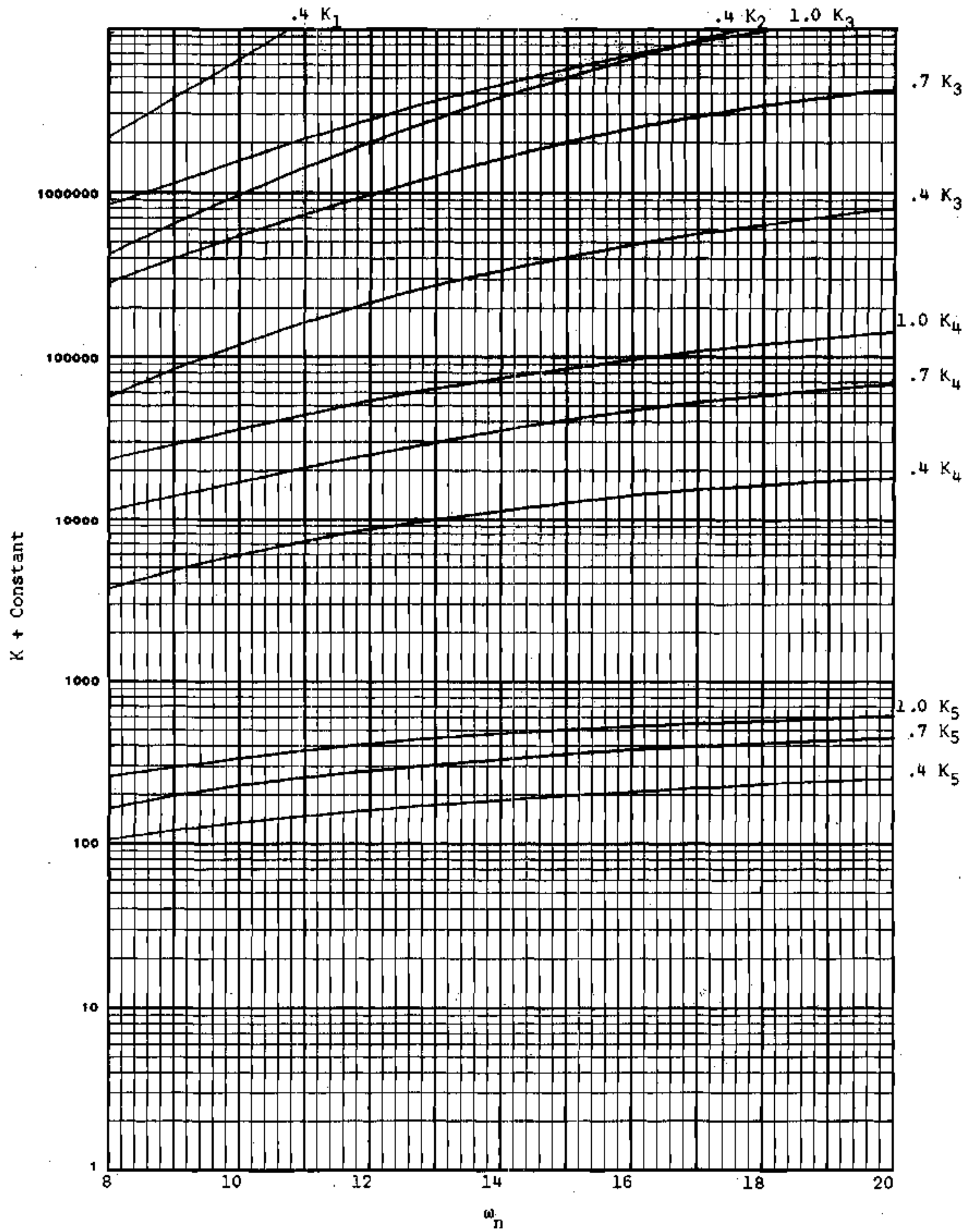


Figure 15b. Fifth Order System
Extra Roots at Ten Times Real Part

degree of equivalency depends upon the distance that the extra roots of the systems have been moved out the real axis.

As a second example, assume a designer has the fifth order system shown in Figure 16.

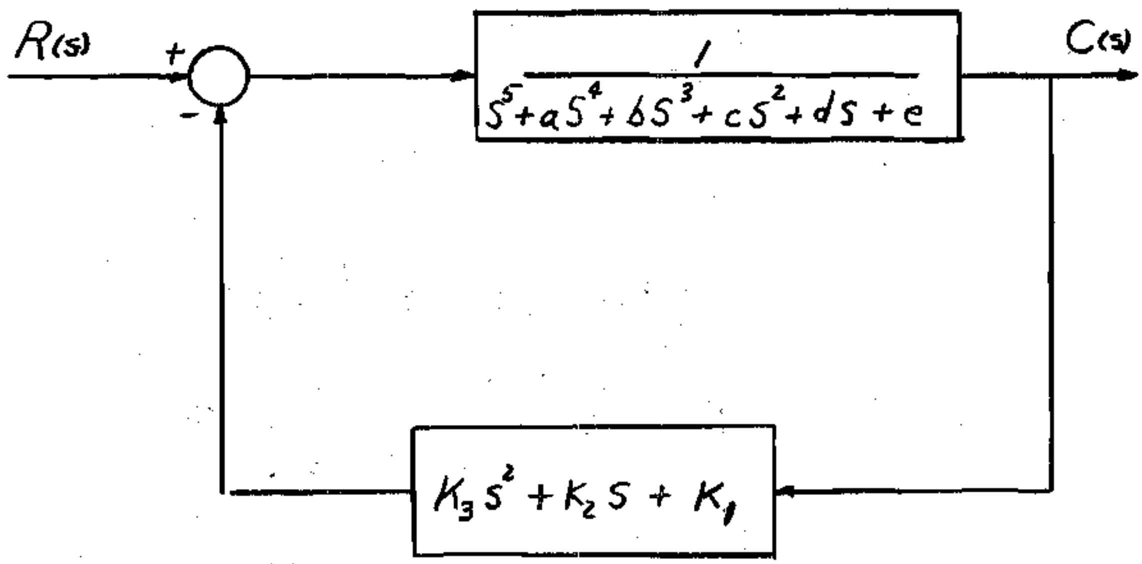


Figure 16. Example Two

Here position, velocity (rate), and acceleration feedback all will be used if necessary (i.e., $|K_i|>0$). The characteristic equation of this system is

$$s^5 + as^4 + bs^3 + (K_3 + c)s^2 + (K_2 + d)s + (K_1 + e) = 0 \qquad (10)$$

If the designer desires a system performance characterized by a second order system with $\zeta = 0.8$ and $\omega_n = 10.0$, then from Figure 13 values of $(K_1 + e) = 1{,}500{,}000$, $(K_2 + d) = 350{,}000$, $(K_3 + c) = 40{,}000$, $(K_4 + b) =$

2,500, and $(K_5 + a) = 86$, may be found to satisfy these conditions. Since K_4 and K_5 equal zero in this system, (a) must equal 86 and (b) must equal 2,500. Since (a) and (b) are defined by the system characteristics, some network synthesis will be necessary to achieve these values. The problem is relieved somewhat in that c, d, and e are not fixed. Their values may change during the synthesis process and they may be allowed to vary up to the values 40,000, 350,000 and 1,500,000, respectively.

Something should be said concerning the accuracy of the results, principally the plotting and computer accuracy and the degree to which the curves of Figures 3, 4, and 5 approach the selected standard. It is felt that the curves are plotted to within 5 per cent of the data obtained from the digital computer. For correlation purposes it was desired to plot all the data for a particular computer "run" on one graph. This necessitated using a wide range of numbers which threaten to destroy the accuracy of the graphs and in essence the value of the graphs. To reduce this tendency each curve was broken into two frequency ranges, 2.0 to 10.0, and 8.0 to 20.0. Periodic checks were made afterwards of particular points to check this accuracy. An accuracy figure of 5 per cent is felt to be conservative.

While no real interpretation of analog and digital computer accuracy was made, they are unquestionably much more accurate than the plotted curves and thus their accuracy need not be considered.

To discuss how well the various curves of Figures 3, 4, and 5 approach the chosen second order system $(s^2 + 2s + 2)$, it might be well to speak in terms of the five time domain specifications recommended by Gibson; delay time, rise time, settling time, per cent overshoot, and final value of error. It is apparent that when a high order system is being made equivalent to a second order system, the degree of equivalency will increase the farther out the negative real axis the extra roots are placed. This is borne out in Figures 3, 4, and 5. Thus, a very good approximation will be achieved if in the third, fourth, and fifth order cases the curves for ten times the real part are used. However, when one then begins to select the (K_i + constant) values from the figures, they are very large, perhaps making physical realization difficult. This is a problem requiring some experience on the part of the designer. He must know what degree of simulation to select, modulated by what magnitude of K_i he can physically achieve.

Using the definitions for the five time domain specifications listed in Appendix C, and placing the extra roots at only three times the real part of the chosen dominant complex pair, there is no significant difference between the values of settling time, per cent overshoot, and final value of error for the standard and for any of the three systems. The greatest difference is in rise time and delay time and here the difference may be tolerable, depending on the system. Using the scales shown for the third order system and the (a) equal three curve, the difference is found to be approximately 0.5 seconds additional rise time, and approximately 0.3 additional seconds for delay. In the fourth order system using the "three times the real part" curve an additional 0.9 seconds is seen for rise time, and approximately 0.7 additional seconds for delay time. In the fifth order system using the three times the real part curve, an additional 1.2 seconds is indicated for rise time and 1.1 seconds more delay time is necessary. The need for moving the roots farther out to more nearly approximate the second order system increases with the order of the actual system or in fact the number of extra roots to be positioned. This is apparent in Figures 3, 4, and 5 when one notes the greater degree of second order approximation achieved by a third order system with extra roots at ten times the real part, compared to a fourth or fifth order system with extra roots at ten times the real part.

In the fourth and fifth order systems as the extra roots were moved, they were kept as complex pairs except in curve eight of the fourth order plot and curve seven of the fifth order plot. Here the roots were made real and an examination of the results indicate very little difference in the curve for the complex pair at three times the real part (curves seven and six, respectively) and the real roots with average absolute value equal to three times the real part. The transient response is seen then to rely more heavily on the real part of the roots than on their imaginary parts. One last gauge perhaps to judge the results by would be the actual damping ratio the system yields as it approaches that of the standard 0.707. From a comparison of Figures 3, 4, and 5 and Figure 5-6 Kuo[20], one might estimate the following results of actual ratios for the curves of Figures 3, 4, and 5.

While many ζ and ω_n values could have been chosen as a standard, it is felt that $\zeta = 0.707$ and $\omega_n = 1.414$ were the likely ones to use. These give the best compromise between fast response and minimum overshoot. These then might be considered a designer's optimum, so that a comparison of actual systems to this standard is deemed a reasonable measure of system merit.

Table 1. Equivalent Damping Ratios

	Third Order			Fourth Order			Fifth Order		
	3X	5X	10X	3X	5X	10X	3X	5X	10X
Equivalent Damping Ratio	0.90	0.75	0.71	0.90	0.80	0.75	2.0	0.90	0.80

In summation, it is felt that the data made available for the determination of the K_i is accurate to approximately 5 per cent. It is also felt by the author that these curves will be of use to system designers, and an examination of the curves in total will give an observer a feel as to the performance that may be expected from certain feedback control systems.

CHAPTER V

CONCLUSIONS

The purpose of this study was to give a control system designer information from which he could select feedback gains to enable him to achieve a desired system performance. The methods used to determined these gains were based on the root locations of the actual system's characteristic equation. As a consequence of this study, several facts became apparent. The following are conclusions reached as a result of this study.

1. The curves in Figures 7 through 15 based on the characteristic equation root locations will aid the feedback control system designer in determining the feedback gains to use to achieve a desired system response.

2. The weight or effect of the several feedback parameters (position, rate, acceleration) varies with the natural frequency; the position parameter having the greatest effect at high frequencies. This is substantiated by observation of the curves in Figures 7 through 15.

3. The need to position the extra roots far out the negative real axis in the s plane in order to more nearly yield second order approximation, increases with the order of the actual system. This is substantiated in Figures 3, 4, and 5. It is recommended that in the third order case the extra root be placed at five times the real part of the chosen complex pair, and in the fourth and fifth order cases at ten times the real part of the chosen complex pair.

CHAPTER VI

RECOMMENDATIONS

While a tremendous amount of work has been done in the field of control system design, many questions are yet to be answered. One important contribution which could be made is the development of a device to measure high derivatives of the output. With an accurate measure of the third derivative of the output (jerk), the designer would be relieved of much of the system synthesis he must do now in trying to eliminate the need for this measurement, thus simplifying the synthesis necessary after using the presented graphs. More accurate acceleration and velocity measuring devices would also aid the system designer.

It is also recommended that a study similar to this one on linear proportional control systems be made on systems of the type shown in Figure 17, where the feedforward transfer function is of the proportional plus derivative type.

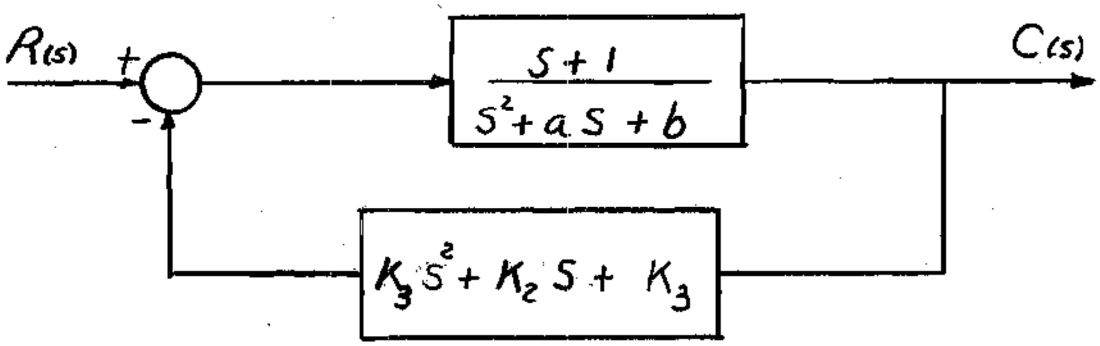


Figure 17. Recommended Problem for Study

Here the characteristic equation is

$$K_1 s^3 + (K_1 + K_2 + 1)s^2 + (K_2 + K_3 + a)s + (K_3 + b) = 0 \qquad (11)$$

If a systematic means could be devised of relating the root locations here to system gains, more varied type problems could be solved.

APPENDICES

APPENDIX A

COMPUTER CIRCUITS USED TO OBTAIN SECOND ORDER APPROXIMATIONS

Figures 18, 19 and 20 show the analog computer circuits used to obtain the plots of Figures 3, 4, and 5, respectively. Once the circuits were set up, then the various curves for the extra root positions were obtained by varying potentiometer settings as necessary to achieve various root locations.

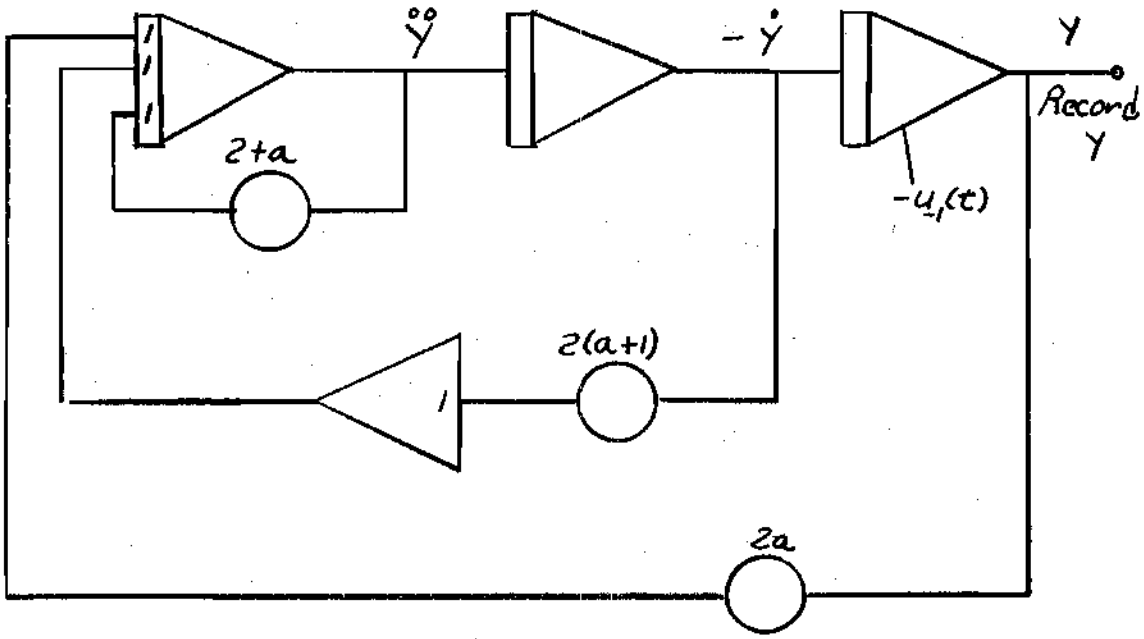


Figure 18. Computer Circuit for Third Order Approximation

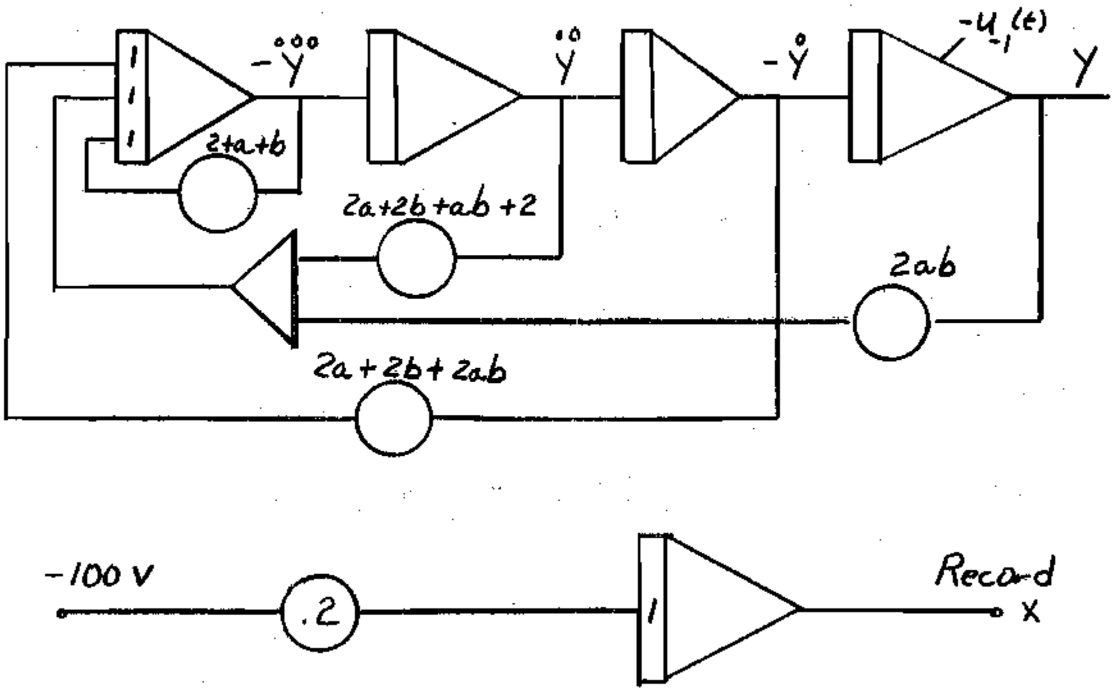
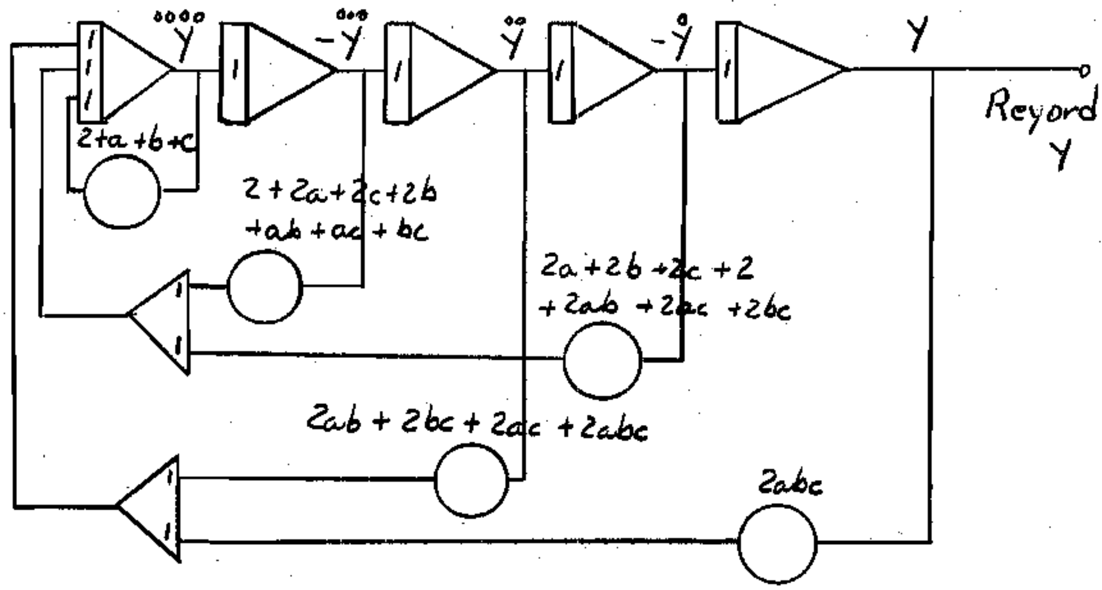


Figure 19. Computer Circuit for Fourth Order Approximation



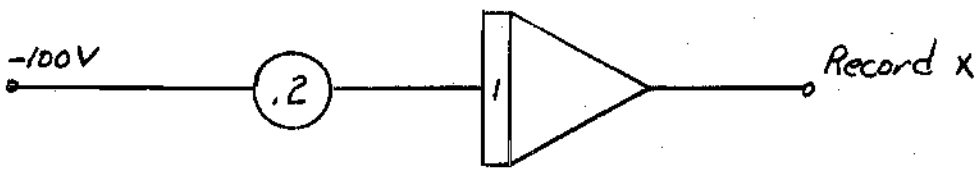


Figure 20. Computer Circuit for Fifth Order Approximation

APPENDIX B

SAMPLE DIGITAL COMPUTER PROGRAM

The Burroughs 220 program presented here depicts an example of the method used to obtain the data plotted in Figures 7 through 15. This particular program is that one used to obtain the data for Figure 7 which is a third order system with its one extra root at three times the real part of the chosen complex pair. Similar programs were used to obtain data for the other curves.

SAMPLE DIGITAL COMPUTER PROGRAM

THE SOLUTION OF FEEDBACK GAINS NECESSARY IN A THIRD ORDER SYSTEM TO SIMULATE SECOND ORDER SYSTEM CHARACTERISTICS

```
2              COMMENT                                                  $
2              WRITE($$TITLE1)                                          $
2              WRITE($$TITLE2)                                          $
2              FOR ZETA=(0.4,0.1,1.0)                                   $
2              FOR WN=(0.1,0.1,10.0),(10.0,1.0,19.0),20.0               $
2   BEGIN      SAM=(((2(ZETA)(WN)(2(ZETA)(WN))@(4(WN)(WN)))/4)          $
2              LTWOR=@(ZETA)(WN))                                       $
2              LTHREER=(@(ZETA)(WN))                                    $
2              KONE=0.0                                                 $
2              LONE=@KONE+LTWOR+LTHREER                                 $
2              IF LONE LSS (@3((2(ZETA)(WN))/2))                        $
2              GO TO HERE                                               $
2              LONE=(@3((2(ZETA)(WN))/2))                               $
2              KONE=@(LONE+LTWOR+LTHREER)                               $
2 HERE         KTWO=LONE(LTWOR+LTHREER)+LTWOR(LTHREER)+(@SAM)           $
2              KTHREE=@(LONE((LTWOR)(LTHREER))+LONE(@SAM))              $
2              WRITE($$ANS,FMT)                 END                     $
2 OUTPUT       ANS(ZETA,WN,LONE,KONE,KTWO,KTHREE)                       $
2 FORMAT       TITLE1(B18,*FEEDBACK GAINS WHEN A THIRD ORDER PLANT
               IS MADE EQUIVALENT TO A CHOSEN SECOND ORDER PLANT*,W2)   $
2 FORMAT       TITLE2(B12,*ZETA*,B8,*WN*,B15,*LONE*,B17,*KONE*,B17,
               *KTWO*,B16,*KTHREE*,W2)                                  $
2 FORMAT       FMT(B10,2S10.5,4S20.8,W0)                                $
```

SAMPLE DIGITAL COMPUTER PROGRAM (Continued)

```
2         FINISH
```

APPENDIX C

CONTROL SYSTEM TERMINOLOGY AND DEFINITIONS (21)(22)

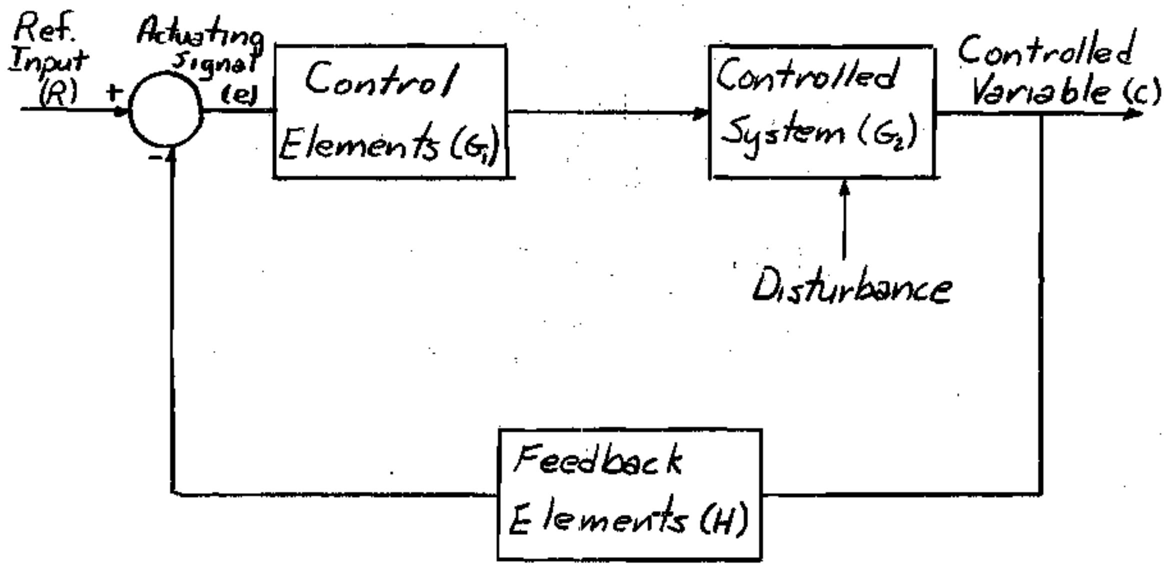


Figure 21. Diagram Illustrating System Terminology

Actuating signal	reference input minus the primary feedback.
Bandwidth	the range of frequencies between zero and the frequency at which the normalized closed-loop transfer function has the magnitude of 0.707, or 3 decibels down.
Control elements	those devices that produce the manipulated variable from the actuating signal.
Controlled system	the device that is to be controlled, frequently a high-power element.
Controlled variable	the quantity that is directly measured and controlled--the output of the controlled system.
Damping ratio	the ratio of the actual damping to the critical value of damping.

Delay time	the time elapsed after application of step input, until the average output reaches half its final value.
Disturbance	any unwanted signal that tends to affect the controlled variable.
Feedback elements	those devices that produce the primary feedback from the controlled variable.
Final value of error	the steady state error as time approaches infinity.
M-peak	the maximum value of the frequency sensitive magnitude of the ratio (output/input) where the input is a sinusoidally varying signal.
Natural frequency (undamped)	the frequency of oscillation of the transient if the damping is zero.
Peak frequency	the frequency in rad/sec at which M-peak occurs.
Peak output inpedance	the maximum value of the magnitude of output impedance z where $z = \frac{\text{output resulting from load disturbance}}{\text{rated load of system}}$
Per cent overshoot	$\frac{(\text{maximum value of response}) - (\text{final value})}{\text{final value}}$
Primary feedback	a signal which is a function of the controlled variable and which is compared with the reference input to obtain the actuating signal.
Reference input	the actual signal input to the system.
Rise time	the time required for the response to rise from 10 to 90 per cent of its final value.
Settling time	the time elapsed after application of a step input until the response of a system falls to, and remains within ± 5 per cent of its final value.

APPENDIX D

RELATIONSHIPS BETWEEN THE ROOTS AND COEFFICIENTS OF POLYNOMIALS OF DEGREE ONE TO FIVE

(where $a = + 1$ and λ = root)

FIRST DEGREE $ax + b = 0$

$b = - \lambda$

SECOND DEGREE $ax^2 + bx + c = 0$

$b = - (\lambda_1 + \lambda_2)$

$c = \lambda_1\lambda_2$

THIRD DEGREE $ax^3 + bx^2 + cx + d = 0$

$b = (\lambda_1 + \lambda_2 + \lambda_3)$

$c = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$

$d = - \lambda_1\lambda_2\lambda_3$

FOURTH DEGREE $ax^4 + bx^3 + cx^2 + dx + e = 0$

$b = - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$

$c = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_1\lambda_4 + \lambda_1\lambda_3 + \lambda_2\lambda_4$

$d = (\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4)$

$e = \lambda_1\lambda_2\lambda_3\lambda_4$

FIFTH DEGREE $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$

$b = - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)$

$c = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_5 + \lambda_1\lambda_5 + \lambda_1\lambda_3$

$$+ \lambda_2\lambda_4 + \lambda_3\lambda_5 + \lambda_2\lambda_5 + \lambda_1\lambda_4$$

$$d = -(\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_5 + \lambda_1\lambda_4\lambda_5 + \lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_3\lambda_5 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_5)$$

$$e = \lambda_1\lambda_2\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4\lambda_5 + \lambda_1\lambda_3\lambda_4\lambda_5 + \lambda_1\lambda_2\lambda_4\lambda_5 + \lambda_1\lambda_2\lambda_3\lambda_5$$

$$f = \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5$$

APPENDIX E

RELATION OF RESPONSE TO LOCATION OF ROOTS (23)

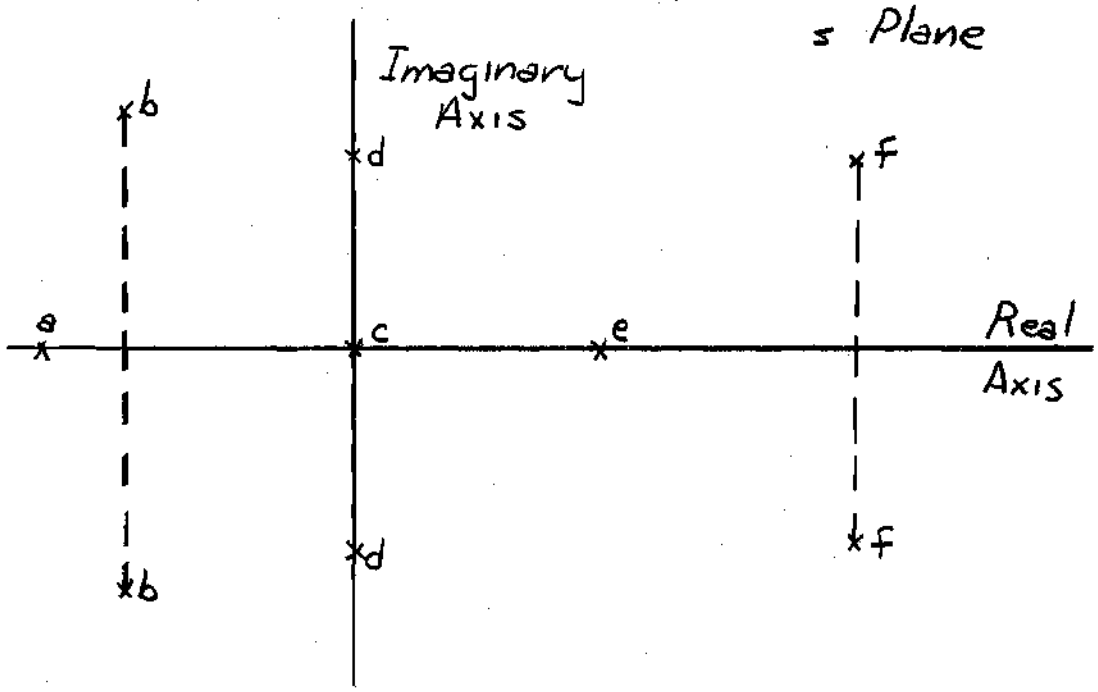


Figure 22. Typical Root Locations

Table 2. Summary of System Response

Position of Pole	Form of Response	Characteristics
a	$Ke^{-\nabla t}$	damped exponential
b	$K_1e^{-\nabla t}(\sin\omega t + \cos\omega t) + K_2$	exponentially damped sinusoid
c	K	constant
d	$K_1 (\sin\omega t + \cos\omega t) + K_2$	constant sinusoid
e	$Ke^{+\nabla t}$	increasing exponential (unstable)
f	$K_1e^{+\nabla t}(\sin\omega t + \cos\omega t) + K_2$	exponentially increasing sinusoid

APPENDIX F

SAMPLE CALCULATIONS

Given: A third order system whose characteristic equation is

$$s^3 + (a + K_3)s^2 + (b + K_2)s + (c + K_1) = 0$$

This system is to be approximated by a second order system defined by a damping ratio of 0.5 and a natural frequency equal to 2.0 rad/sec. The extra root is to be placed at three times the real part of the complex pair of roots.

Required: $a + K_3$

$b + K_2$

$c + K_1$

Solution: With $\zeta = 0.5$ and $\omega_n = 2.0$ rad/sec the second order system is found to be

$$s^2 + 2\zeta\omega_n{}^2 = 0$$

$$s^2 + 2s + 4 = 0 \qquad \text{with roots}$$

$$- 1.0 \pm j\sqrt{3}$$

therefore, we can let

$$\lambda_1 = -1.0 + j\sqrt{3}$$

$$\lambda_2 = -1.0 - j\sqrt{3}$$

$$\lambda_3 = -3.0$$

From Appendix D letting

$$b = (a + K_3)$$

$$c = (b + K_2)$$

$$d = (c + K_1)$$

then $$a + K_3 = -\left[(-1.0 + j\sqrt{3}) + (-1.0 - j\sqrt{3}) + (-3.0)\right]$$

$$b + K_2 = \left[(-1.0 + j\sqrt{3})(-1.0 - j\sqrt{3}) + (-3.0)(-1.0 - j\sqrt{3}) + (-3.0)(-1.0 + j\sqrt{3})\right]$$

$$c + K_1 = -\left[(-1.0 + j\sqrt{3})(-1.0 - j\sqrt{3})(-3.0)\right]$$

therefore,

$$a + K_3 = 5$$

$$b + K_2 = 10$$

$$c + K_1 = 12$$

which checks the values from Figure 7a.

BIBLIOGRAPHY

LITERATURE CITED

1. Smith, C., *A Treatise on Algebra,* MacMillan and Company Limited, London, 1933, pp. 124-125.

2. D'Azzo, J. J., and Houpis, C. H., *Feedback Control System Analysis and Synthesis,* McGraw-Hill Book Company, Inc., New York, New York, 1960, p. 73.

3. Gibson, J. E., "Set of Standard Specifications for Linear Automatic Control Systems," *Transactions of the American Institute of Electrical Engineers,* Part II, Vol. 80, 1961, pp. 65-77.

4. Langill, A. W., "Root-Locus Systems Analysis," *Electro-Technology,* October, 1963, pp. 80-94.

5. D'Azzo, J. J., and Houpis, C. H., *op. cit.,* p. 130.

6. Evans, W. R., "Graphical Analysis of Control Systems," *Transactions of the American Institute of Electrical Engineers,* Part II, Vol. 67, 1948, pp. 547-551.

7. Steiglitz, K., "An Analytical Approach to Root Loci," *Transactions of the Institute of Radio Engineers,* Part 2, 1961, pp. 326-332.

8. Jawor, T., "Using the Root Locus," *Control Engineering,* October, 1959, pp. 119-122.

9. Matkowsky, B. J., and Zemanian, A. H., "A Contribution to Root-Locus Techniques for the Analysis of Transient Responses," *Transactions of the Institute of Radio Engineers on Automatic Control,* April, 1962, pp. 69-73.

10. Elgerd, O. I., and Stephens, W. C., "Effect on Closed-Loop Transfer Function Pole and Zero Locations on the Transient Response of Linear Control Systems," *Transactions of the American Institute of Electrical Engineers,* Part II, Vol. 78, 1959, pp. 121-127.

11. Mitrovic, D., "Graphical Analysis and Synthesis of Feedback Control Systems," *Transactions of the American Institute of Electrical Engineers,* Part II, Vol. 77, 1958, p. 476-502.

12. Mariotti, F., "A Direct Method of Compensating Linear Feedback Systems," *Transactions of the American Institute of Electrical Engineers,* Part II, Vol. 79, 1960, pp. 527-538.

13. Numakura, T., and Miura, T., "A New Stability Criterion of Linear Servomechanisms by a Graphical Method," *Transactions of the American Institute of Electrical Engineers*, Part II, Vol. 76, 1957, pp. 40-48.

14. Lendaris, G. G., and Smith, O. J. M., "Complex-Zero Signal Generator for Rapid System Testing," *Transactions of the American Institute of Electrical Engineers*, Part II, Vol. 77, 1958, pp. 534-539.

15. Brussolo, J. A., "Pole Determinations with Complex-Zero Inputs," *Institute of Radio Engineers WESCON Convention Record*, Part 4, 1959, pp. 154-170.

16. Truxal, J. G., *Automatic Feedback Control System Synthesis*, McGraw-Hill Book Company, Inc., New York, New York, 1955, 675 p.

17. Gibson, J. E., *Nonlinear Automatic Control*, McGraw-Hill Book Company, Inc., New York, New York, 1963, 585 p.

18. Thaler, G. J., and Brown, R. G., *Analysis and Design of Feedback Control Systems*, McGraw-Hill Book Company, Inc., 1960, 648 p.

19. Savant, C. J., *Control System Design*, McGraw-Hill Book Company, Inc., New York, New York, 1964, 457 p.

20. Kuo, B. C., *Automatic Control Systems*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962, 504 p.

21. D'Azzo, J. J., and Houpis, C. H., *op. cit.*, pp. 505-507.

22. Gibson, J. E., *op. cit.*, pp. 65-77.

23. D'Azzo, J. J., and Houpis, C. H., *op. cit.*, p. 77.

OTHER REFERENCES

Billingsley, D. S., and Rekoff, M. G., "Evaluation of Transient Response Coefficients," *Transactions of the Institute of Radio Engineers on Automatic Control*, February, 1961, pp. 80-83.

Blackman, P. F., "Pole Zero Approach to System Analysis," *Control*, November 1960-August 1961.

Campbell, W. G., *Form and Style in Thesis Writing*, Houghton Mifflin Company, New York, New York, 1954, 114 p.

Chen, K., "A Quick Method for Estimating Closed-Loop Poles of Control Systems," *Transactions of the American Institute of Electrical Engineers*, Part II, Vol. 76, 1957, pp. 80-87.

DeMello, F. P., "Evaluation of Transient System Response," *Transactions of the American Institute of Electrical Engineers*, Part II, Vol. 78, 1959, pp. 176-186.

Koenig, J. F., "Two theorems on the Number of Real Roots of the Characteristic Equation of Any Stable Linear Physical System," *Transactions of the American Institute of Electrical Engineers*, Part II, Vol. 77, 1958, pp. 670-673.

Huang, R. Y., "The Sensitivity of the Poles of Linear Closed-Loop Systems," *Transactions of the American Institute of Electrical Engineers*, Part II, Vol. 77, 1958, pp. 182-186.

Mulligan, J. H., "The Effect of Pole and Zero Location on the Transient Response of Linear Dynamic Systems," *Institute of Radio Engineers Proceedings*, May, 1949, pp. 516-529.

Raven, F. H., *Automatic Control Engineering*, McGraw-Hill Book Company, Inc., New York, New York, 1961, 402 p.

Ritow, I., "Automatic Control System Design," *Electrical Manufacturing*, April-September, 1959.

Thaler, G. J., *Elements of Sevvomechanism Theory*, McGraw-Hill Book Company, Inc., New York, New York, 1955, 282 p.

Yeh, V. C. M., "Synthesis of Feedback Control Systems," *Transactions of the American Institute of Electrical Engineers*, Part II, Vol. 75, 1956, pp. 85-89.